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Optimal regularity and exponential stability for the Blackstock–Crighton equation in L_p-spaces with Dirichlet and Neumann boundary conditions

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Abstract. The Blackstock–Crighton equation models nonlinear acoustic wave propagation in monatomic gases. In the present work, we investigate the associated inhomogeneous Dirichlet and Neumann boundary value problems in a bounded domain and prove long-time well-posedness and exponential stability for sufficiently small data. The solution depends analytically on the data. In the Dirichlet case, the solution decays to zero and the same holds for Neumann conditions if the data have zero mean. We choose an optimal L_p-setting, where the regularity of the initial and boundary data is necessary and sufficient for existence, uniqueness and regularity of the solution. The linearized model with homogeneous boundary conditions is represented as an abstract evolution equation for which we show maximal L_p-regularity. In order to eliminate inhomogeneous boundary conditions, we establish a general higher regularity result for the heat equation. We conclude that the linearized model induces a topological linear isomorphism and then solves the nonlinear problem by means of the implicit function theorem.

1. Introduction

An acoustic wave propagates through a medium as a local pressure change. Nonlinear effects typically occur in case of acoustic waves of high amplitude which are used for several medical and industrial purposes such as lithotripsy, thermotherapy, ultrasound cleaning or welding and sonochemistry. Research on mathematical aspects of nonlinear acoustic wave propagation is therefore not only interesting from a mathematicians point of view. In fact, enhancement of the mathematical understanding of the underlying models may help to solve problems arising in various applications.

The present work provides an analysis of the Blackstock–Crighton–Kuznetsov equation

$$(a\Delta - \partial_t)\left(u_{tt} - c^2\Delta u - b\Delta u_t\right) = \left(\tfrac{1}{c^2}\tfrac{B}{2A}(u_t)^2 + |\nabla u|^2\right)_{tt} \tag{1.1}$$

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and the Blackstock–Crighton–Westervelt equation

$$(a\Delta - \partial_t)\left(u_{tt} - c^2\Delta u - b\Delta u_t\right) = \left(\tfrac{1}{c^2}\left(1 + \tfrac{B}{2A}\right)(u_t)^2\right)_{tt} \tag{1.2}$$

for the acoustic velocity potential u, where c is the speed of sound, b is the diffusivity of sound and a is the heat conductivity of the fluid. Note that $a = \nu \mathrm{Pr}^{-1}$, where ν is the kinematic viscosity and Pr denotes the Prandtl number. The quantity B/A is known as the parameter of nonlinearity and is proportional to the ratio of the coefficients of the quadratic and linear terms in the Taylor series expansion of the variations of the pressure in a medium in terms of variations of the density, see [17, Section 4.2].

Equations (1.1) and (1.2) are derived from the full equations of motion for thermoviscous fluids by a small modification of Blackstock's approximation [4] and using Becker's assumption for monatomic gases which corresponds to the choice $\mu_B = 0$ and $\mathrm{Pr} = 3/4$ for the bulk viscosity and the Prandtl number, respectively. The acoustic diffusivity then becomes $b = 4\gamma\nu/3$, where γ is the specific heat ratio. Moreover, in case of monatomic gases we have $B/A = \gamma - 1$. Alternatively, (1.1) and (1.2) can be expressed in terms of the acoustic pressure p via the pressure density relation $\rho u_t = p$, where ρ denotes the mass density. Note that (1.2) is obtained from (1.1) by neglecting local nonlinear effects in the sense that the expression $c^2|\nabla u|^2 - (u_t)^2$ is sufficiently small. With $k = c^{-2}(1 + B/2A)$, we represent (1.2) as

$$(-1 - 2ku_t)u_{ttt} + (a+b)\Delta u_{tt} + c^2\Delta u_t - ab\Delta^2 u_t - ac^2\Delta^2 u = 2k(u_{tt})^2,$$

which shows that potential degeneracy is an important feature of (1.1) and (1.2). We avoid this degeneracy by considering solutions for sufficiently small initial and boundary data.

For a rigorous derivation of the equations under consideration, we refer to [5, Chapter 1] and [7], whereas a detailed introduction to the theory and applications of nonlinear acoustics is provided by [17]. Summing up, we here just emphasize that (1.1) and (1.2) are approximate equations governing finite amplitude sound in monatomic gases (e.g., helium, xenon, argon).

While (1.1) and (1.2) are enhanced models in nonlinear acoustics, the Kuznetsov

$$u_{tt} - b\Delta u_t - c^2\Delta u = \left(\tfrac{1}{c^2}\tfrac{B}{2A}(u_t)^2 + |\nabla u|^2\right)_t \tag{1.3}$$

and the Westervelt equation

$$u_{tt} - b\Delta u_t - c^2\Delta u = \left(\tfrac{1}{c^2}\left(1 + \tfrac{B}{2A}\right)(u_t)^2\right)_t \tag{1.4}$$

are classical, well accepted and widely used models governing sound propagation in fluids. As (1.1) and (1.2), they are derived from the basic equations in fluid mechanics. The Kuznetsov equation is the more general one of these classical models; in particular, the Westervelt equation is obtained from the Kuznetsov equation by neglecting local nonlinear effects. Moreover, for a small ratio of ν and Pr, that is, for small heat conductivity, (1.3) and (1.4) can be regarded as simplifications of (1.1) and (1.2), respectively.

The classical models (1.3) and (1.4) have recently been extensively investigated. In particular, results on well-posedness for the Kuznetsov and the Westervelt equation with homogeneous Dirichlet [19] and inhomogeneous Dirichlet [22], [21] and Neumann [20] boundary conditions have been shown in an $L_2(\Omega)$-setting on spatial domains $\Omega \subset \mathbb{R}^n$ of dimension $n \in \{1, 2, 3\}$. Moreover, there are results on optimal regularity and long-time behavior of solutions for the Westervelt equation with homogeneous Dirichlet [26] and for the Kuznetsov equation with inhomogeneous Dirichlet [27] boundary conditions in $L_p(\Omega)$-spaces, where the spatial domain $\Omega \subset \mathbb{R}^n$ is of arbitrary dimension.

On the contrary, mathematical research on higher-order partial differential equations arising in nonlinear acoustics is still in an early stage. Well-posedness and exponential decay results for the homogeneous Dirichlet boundary value problems associated with (1.1) and (1.2) in an $L_2(\Omega)$-setting where $\Omega \subset \mathbb{R}^n$, $n \in \{1, 2, 3\}$, have been shown in [6] and [8], respectively. In the present work, we consider (1.1) and (1.2) with inhomogeneous Dirichlet and Neumann boundary conditions in $L_p(\Omega)$-spaces, where the spatial domain Ω is of dimension $n \in \mathbb{N}$. We show global well-posedness and long-time behavior of solutions in an optimal functional analytic setting in the sense that the regularity of the initial and boundary data is both necessary and sufficient for the regularity of the solution. While in [6,8] the results were proved by means of energy estimates and Banach's fixed-point theorem, the techniques used in the present paper are based on maximal L_p-regularity and the implicit function theorem.

We suppose that $\Omega \subset \mathbb{R}^n$, $n \in \mathbb{N}$, is a bounded smooth domain, i.e., an open, connected and bounded subset of $\mathbb{R}^n$ with smooth boundary Γ. Let $J = (0, T)$ for some finite $T > 0$ or $J = \mathbb{R}_+ = (0, \infty)$. We consider the inhomogeneous Dirichlet boundary value problem

$$\begin{cases} (a\Delta - \partial_t)(u_{tt} - b\Delta u_t - c^2\Delta u) = (k(u_t)^2 + s|\nabla u|^2)_{tt} & \text{in } J \times \Omega, \\ (u, \Delta u) = (g, h) & \text{on } J \times \Gamma, \\ (u, u_t, u_{tt}) = (u_0, u_1, u_2) & \text{on } \{t = 0\} \times \Omega, \end{cases} \tag{1.5}$$

and the inhomogeneous Neumann boundary value problem

$$\begin{cases} (a\Delta - \partial_t)(u_{tt} - b\Delta u_t - c^2\Delta u) = (k(u_t)^2 + s|\nabla u|^2)_{tt} & \text{in } J \times \Omega, \\ (\partial_\nu u, \partial_\nu \Delta u) = (g, h) & \text{on } J \times \Gamma, \\ (u, u_t, u_{tt}) = (u_0, u_1, u_2) & \text{on } \{t = 0\} \times \Omega, \end{cases} \tag{1.6}$$

where $u_0, u_1, u_2 \colon \Omega \to \mathbb{R}$ and $g, h \colon J \times \Gamma \to \mathbb{R}$ are given, $u \colon J \times \Omega \to \mathbb{R}$ is the unknown, $u(t, x)$, and $a, b, c > 0$ and $k \in \mathbb{R}$ are positive constants. Moreover, $\partial_\nu u = \nu \cdot \nabla u|_\Gamma$ denotes the normal derivative of u in terms of the outer unit normal vector ν. The parameter $s \in \{0, 1\}$ allows us to switch between (1.1) and (1.2).

We point out that the present work extends the results from [6] in several ways. First, while in [6] the Blackstock–Crighton equation was considered with homogeneous Dirichlet boundary conditions, we also allow for inhomogeneous Dirichlet as well as Neumann boundary conditions. We are able to remove the restriction $n \in \{1, 2, 3\}$ on the dimension of the spatial domain Ω. Instead of $L_2(\Omega)$, we consider (1.1) and (1.2) in $L_p(\Omega)$ where $p \in (1, \infty)$ in case of the linearized equation and $p > \max\{n/4 + 1/2, n/3\}$ in case of the nonlinear equations (1.5) and (1.6). In particular, we require $p \in (5/4, \infty)$ in case $n = 3$, and then, $p = 2$ is admissible. Moreover, most notably, our conditions on the regularity of the data (g, h, u_0, u_1, u_2) are necessary and sufficient for the existence of a unique solution of the Blackstock–Crighton equation within a certain regularity class of $L_p(J \times \Omega)$.

Our strategy for solving (1.5) and (1.6) is to prove that their linearizations induce isomorphisms between suitable Banach spaces and to apply the implicit function theorem. In some sense, these linearizations can be considered as a composition of a heat problem and another linearized problem for the Westervelt equation. While the linearized Westervelt equation can be handled similar as in [26,27], the heat equation has to be solved with higher regularity conditions.

The paper is organized as follows. In Sect. 2 we recall several preliminaries and facts for analyzing (1.5) and (1.6). Section 3 is devoted to the inhomogeneous Dirichlet boundary value problem (1.5) for which we prove global well-posedness (Theorem 3.6) and exponential stability (Corollary 3.7). In Sect. 4, we treat the inhomogeneous Neumann boundary value problem (1.6) and prove its well-posedness in Theorem 4.6. At first we get local well-posedness, but if the data have zero mean, then also global well-posedness holds for (1.6). Moreover, Corollary 4.8 states long-time behavior of solutions.

Appendix A is devoted to the temporal trace operator acting on a class of anisotropic Sobolev spaces. We present its mapping properties, provide a right-inverse and construct functions with prescribed higher-order initial data. In Appendix B, we prove some higher regularity results for the heat equation with inhomogeneous Dirichlet or Neumann boundary conditions and inhomogeneous initial conditions in a far more general framework than needed in the main text. We explicitly state all necessary compatibility conditions between initial and boundary data and show how these are used to construct a solution with high regularity.

2. Preliminaries

The purpose of this section is to introduce the notation and to recall several important facts and results we need to prove global well-posedness and long-time behavior of solutions for (1.5) and (1.6). As already mentioned in Sect. 1, we always assume that $\Omega \subset \mathbb{R}^n$, $n \in \mathbb{N}$, is a bounded domain with smooth boundary $\Gamma = \partial\Omega$. We write J for a time interval and consider either $J = (0, T)$ for some finite $T > 0$ or $J = \mathbb{R}_+ = (0, \infty)$.

2.1. Function spaces, operators, embeddings and traces

The space $BUC^k(\Omega)$ contains all k-times Fréchet differentiable functions $\Omega \to \mathbb{R}$, whose derivatives up to order k are bounded and uniformly continuous. For $p \in (1,\infty)$, let $L_p(\Omega)$ denote the space of (equivalence classes of) Lebesgue measurable p-integrable functions $\Omega \to \mathbb{R}$. We write $W_p^m(\Omega)$ for the Sobolev–Slobodeckij space and $H_p^m(\Omega)$ for the Bessel potential space of order $m \in [0,\infty)$, where we have $W_p^m(\Omega) = H_p^m(\Omega)$ if $m \in \mathbb{N}_0$. For $p \in [1,\infty)$, $q \in [1,\infty]$, $s \in \mathbb{R}_+$, the Besov space $B_{p,q}^s(\Omega)$ is defined as $(L_p(\Omega), W_p^m(\Omega))_{s/m,q}$ where $m = \lceil s \rceil$ and $(\cdot,\cdot)_{s/m,q}$ indicates real interpolation. It holds that $B_{p,p}^s(\Omega) = W_p^s(\Omega)$ if $s \in \mathbb{R}_+ \backslash \mathbb{N}$ and $B_{p,q}^s(\Omega) = W_p^s(\Omega)$ if $p = q = 2$. Moreover,

$$B_{p,q}^s(\Omega) = (W_p^k(\Omega), W_p^m(\Omega))_{\Theta,q}, \tag{2.1}$$

where $0 \le k < s < m$ and $s = (1-\Theta)k + \Theta m$. We also let $W_p^s(\Omega; X)$, $H_p^s(\Omega; X)$ and $B_{p,q}^s(\Omega; X)$ denote the vector-valued versions of these spaces. The space ${}_0W_p^s(\mathbb{R}_+; X)$ is the closure of $C_c^\infty(\mathbb{R}_+; X)$ in $W_p^s(\mathbb{R}_+; X)$. For $k < s - 1/p < k+1$ with $k \in \mathbb{N}_0$, it consists precisely of those functions with vanishing initial traces $\partial_t^j u(0) = 0$ for $0 \le j \le k$ [3, Theorem 4.7.1].

We always write $X \hookrightarrow Y$ if the Banach space X is continuously embedded into the Banach space Y. Moreover, let $L(X,Y)$ be the space of all bounded linear operators between X and Y. A linear operator $A\colon X \to Y$ is called an isomorphism if it is bounded and bijective. Then the closed graph theorem implies that $A^{-1}\colon Y \to X$ is also bounded, and therefore, $A\colon X \to Y$ is a homeomorphism. Now, let X and $\mathbb{X}$ be Banach spaces such that $\mathbb{X} \hookrightarrow L_{1,\mathrm{loc}}(J; X)$ where $L_{1,\mathrm{loc}}(J; X)$ is the space of locally integrable functions $J \to X$. For any $\omega \in \mathbb{R}$, we define the exponentially weighted space

$$\mathrm{e}^{\omega}\mathbb{X} = \{u \in L_{1,\mathrm{loc}}(J; X)\colon \mathrm{e}^{-\omega t} u \in \mathbb{X}\},$$

equipped with the norm $\|u\|_{\mathrm{e}^{\omega}\mathbb{X}} = \|\mathrm{e}^{-\omega t} u\|_{\mathbb{X}}$, where $\mathrm{e}^{-\omega t} u$ denotes the mapping $[t \mapsto \mathrm{e}^{-\omega t} u(t)]$.

Let $-\Delta_D\colon \mathcal{D}(\Delta_D) \to L_p(\Omega)$, $u \mapsto -\Delta u$ denote the negative Dirichlet Laplacian with domain $\mathcal{D}(\Delta_D) = \{u \in W_p^2(\Omega)\colon u = 0 \text{ on } \Gamma\}$. We recall that the spectrum $\sigma(-\Delta_D)$ is a discrete subset of $(0,\infty)$ consisting only of eigenvalues $\lambda_n^D = \lambda_n(-\Delta_D)$, $n \in \mathbb{N}_0$ with finite multiplicity. We write $\lambda_0^D > 0$ for the smallest eigenvalue of $-\Delta_D$. Moreover, the negative Neumann Laplacian $-\Delta_N\colon \mathcal{D}(\Delta_N) \to L_p(\Omega)$, $u \mapsto -\Delta u$ with domain $\mathcal{D}(\Delta_N) = \{u \in W_p^2(\Omega)\colon \partial_\nu u = 0 \text{ on } \Gamma\}$ has a discrete spectrum $\sigma(-\Delta_N) \subset [0,\infty)$ which contains only eigenvalues $\lambda_n^N = \lambda_n(-\Delta_N)$, $n \in \mathbb{N}_0$, of finite multiplicity. Here, $\lambda_0^N = 0$ is an isolated point of $\sigma(-\Delta_N)$ which can be removed when introducing the space $L_{p,0}(\Omega) = \{u \in L_p(\Omega)\colon \int_\Omega u \,\mathrm{d}x = 0\}$ and considering $-\Delta_{N,0}\colon \mathcal{D}(\Delta_{N,0}) \to L_{p,0}(\Omega)$, $u \mapsto -\Delta u$ with $\mathcal{D}(\Delta_{N,0}) = \mathcal{D}(\Delta_N) \cap L_{p,0}(\Omega)$. We then have $\sigma(-\Delta_{N,0}) \subset (0,\infty)$ where $\lambda_1^N = \lambda_1(-\Delta_N) > 0$ is the smallest eigenvalue of $-\Delta_{N,0}$. For details, we refer to [5, Section 4.1].

We shall use the embeddings $W^1_p(J) \hookrightarrow BUC(J)$ and $W^s_p(\Omega) \hookrightarrow W^t_p(\Omega)$ for $s \geq t$. We always write $\gamma_D = \cdot|_\Gamma$ and $\gamma_N = \partial_\nu \cdot |_\Gamma = \nu \cdot (\nabla \cdot)|_\Gamma$ for the Dirichlet and the Neumann trace, respectively. Moreover, $\gamma_t = \cdot|_{t=0}$ denotes the temporal trace. Let $B \in \{D, N\}$, $j_D = 0$, $j_N = 1$. For $p \in (1, \infty)$, $k \in \mathbb{N}$ and $l \in \mathbb{N}_0$, the spatial trace

$$\begin{aligned} u \mapsto \gamma_B u \colon & W^{k+l}_p(J; L_p(\Omega)) \cap W^k_p(J; W^{2l}_p(\Omega)) \\ & \to W^{k+l-j_B/2-1/2p}_p(J; L_p(\Gamma)) \cap W^k_p(J; W^{2l-j_B-1/p}_p(\Gamma)) \end{aligned} \tag{2.2}$$

is bounded, see Lemma B.6. Furthermore, the trace

$$u \mapsto \gamma_B u \colon W^s_p(\Omega) \to B^{s-j_B-1/p}_{p,p}(\Gamma) \tag{2.3}$$

is bounded for every $s \in (j_B + 1/p, \infty)$, cf. [30, Theorem 3.3.3]. The temporal trace

$$\gamma_t \colon u \mapsto u|_{t=0} \colon W^\alpha_p(J; W^s_p(\Omega)) \cap L_p(J; W^{s+2\alpha}_p(\Omega)) \to B^{s+2\alpha-2/p}_{p,p}(\Omega) \tag{2.4}$$

is bounded for $\alpha \in (1/p, 1]$ and $s \in [0, \infty)$, unless either s or $s + 2\alpha$ is an integer. Finally, thanks to [28, Proposition 3.2] and [12, Lemma 2.61, Proposition 2.75], we may employ the mixed derivative embeddings

$$H^{t+\tau}_p(J; H^s_p(\Omega)) \cap H^t_p(J; H^{s+\sigma}_p(\Omega)) \hookrightarrow H^{t+\theta\tau}_p(J; H^{s+(1-\theta)\sigma}_p(\Omega)), \tag{2.5a}$$

$$B^{t+\tau}_{p,p}(J; H^s_p(\Omega)) \cap H^t_p(J; B^{s+\sigma}_{p,p}(\Omega)) \hookrightarrow B^{t+\theta\tau}_{p,p}(J; H^{s+(1-\theta)\sigma}_p(\Omega)), \tag{2.5b}$$

$$B^{t+\tau}_{p,p}(J; H^s_p(\Omega)) \cap H^t_p(J; B^{s+\sigma}_{p,p}(\Omega)) \hookrightarrow H^{t+\theta\tau}_p(J; B^{s+(1-\theta)\sigma}_{p,p}(\Omega)), \tag{2.5c}$$

where $p \in (1, \infty)$, $t, s \in [0, \infty)$, $\tau, \sigma \in (0, \infty)$, $\theta \in (0, 1)$. The embeddings (2.4) and (2.5) remain valid when Ω is replaced by its boundary Γ and they are frequently used for checking the continuity of differential operators in anisotropic spaces.

2.2. Maximal L_p-regularity

Let $J = (0, T)$ or $J = \mathbb{R}_+ = (0, \infty)$ and assume $p \in (1, \infty)$. We say that a closed linear operator $A \colon \mathcal{D}(A) \to X$ with dense domain $\mathcal{D}(A)$ in a Banach space X admits maximal L_p-regularity on J if for each $F \in L_p(J; X)$ the Cauchy problem

$$v_t(t) + Av(t) = F(t),\ t \in J, \qquad v(0) = v_0, \tag{2.6}$$

admits a unique solution $u \in \mathbb{E}(J) = W^1_p(J; X) \cap L_p(J; \mathcal{D}(A))$ for $v_0 = 0$.

Furthermore, the inhomogeneous Cauchy problem (2.6) is said to admit maximal L_p-regularity, if

$$(\partial_t + A, \gamma_t) \colon \mathbb{E}(J) \to L_p(J; X) \times \gamma_t \mathbb{E}(J), \qquad v \mapsto (F, v_0) \tag{2.7}$$

is a homeomorphism. Then its inverse is the solution map

$$(\partial_t + A, \gamma_t)^{-1} \colon L_p(J; X) \times \gamma_t \mathbb{E}(J) \to \mathbb{E}(J), \qquad (F, v_0) \mapsto v. \tag{2.8}$$

If $A \colon \mathcal{D}(A) \to X$ has maximal L_p-regularity on J, then the Cauchy problem (2.6) has maximal L_p-regularity on J, cf. Section III.1.5 in [2]. The following result is very useful and will be used several times throughout this paper.

LEMMA 2.1. (cf. [2, Proposition III.1.5.3]) *Let $\alpha \in \mathbb{R}$. Suppose that*

$$(\partial_t + \alpha + A, \gamma_t) \colon \mathbb{E}(J) \to L_p(J; X) \times \gamma_t \mathbb{E}(J)$$

is a homeomorphism. Then

$$(\partial_t + A, \gamma_t) \colon \mathrm{e}^{\alpha} \mathbb{E}(J) \to \mathrm{e}^{\alpha} L_p(J; X) \times \gamma_t \mathbb{E}(J)$$

is a homeomorphism.

2.3. Optimal regularity results

In order to prove optimal regularity for the linearized versions of (1.5) and (1.6), we need results on optimal regularity for the heat equation and the linearized Westervelt equation. First we consider Dirichlet boundary conditions. Recall that $\lambda_0^D > 0$ always denotes the smallest eigenvalue of the negative Dirichlet Laplacian in $L_p(\Omega)$. In the following, we always assume $a, b, c \in (0, \infty)$.

LEMMA 2.2. ([24, Proposition 8]) *Let $p \in (1, \infty)$ and $\omega \in (0, a\lambda_0^D)$. Then the initial boundary value problem for the heat equation*

$$\begin{cases} u_t - a\Delta u = f & \text{in } \mathbb{R}_+ \times \Omega, \\ u = g & \text{on } \mathbb{R}_+ \times \Gamma, \\ u = u_0 & \text{on } \{t = 0\} \times \Omega, \end{cases} \tag{2.9}$$

has a unique solution

$$u \in \mathrm{e}^{-\omega} \mathbb{H}_u, \qquad \mathbb{H}_u = W_p^1(\mathbb{R}_+; L_p(\Omega)) \cap L_p(\mathbb{R}_+; W_p^2(\Omega)),$$

if and only if the given data f, g and u_0 satisfy the regularity conditions

(i) $f \in \mathrm{e}^{-\omega} L_p(\mathbb{R}_+ \times \Omega)$,
(ii) $u_0 \in W_p^{2-2/p}(\Omega)$,
(iii) $g \in \mathrm{e}^{-\omega} \mathbb{H}_\Gamma$, $\mathbb{H}_\Gamma = W_p^{1-1/2p}(\mathbb{R}_+, L_p(\Gamma)) \cap L_p(\mathbb{R}_+; W_p^{2-1/p}(\Gamma))$,
(iv) $u_0|_\Gamma = g|_{t=0}$ *in the sense of traces.*

LEMMA 2.3. ([27, Lemma 5]) *Suppose $p \in (1, \infty) \backslash \{3/2\}$ and define $\omega_0 = \min\{b\lambda_0^D/2, c^2/b\}$. Then for every $\omega \in (0, \omega_0)$ there exists a unique solution*

$$u \in \mathrm{e}^{-\omega} \mathbb{W}_u, \qquad \mathbb{W}_u = W_p^2(\mathbb{R}_+; L_p(\Omega)) \cap W_p^1(\mathbb{R}_+; W_p^2(\Omega)),$$

of the linear initial boundary value problem

$$\begin{cases} u_{tt} - b\Delta u_t - c^2 \Delta u = f, & \text{in } \mathbb{R}_+ \times \Omega, \\ u = g, & \text{on } \mathbb{R}_+ \times \Gamma, \\ (u, u_t) = (u_0, u_1) & \text{on } \{t = 0\} \times \Omega, \end{cases} \tag{2.10}$$

if and only if the data satisfy the following conditions:

(i) $f \in \mathrm{e}^{-\omega} L_p(\mathbb{R}_+ \times \Omega)$,
(ii) $u_0 \in W_p^2(\Omega)$, $u_1 \in W_p^{2-2/p}(\Omega)$,
(iii) $g \in \mathrm{e}^{-\omega}\mathbb{W}_\Gamma$, $\mathbb{W}_\Gamma = W_p^{2-1/2p}(\mathbb{R}_+; L_p(\Gamma)) \cap W_p^1(\mathbb{R}_+; W_p^{2-1/p}(\Gamma))$,
(iv) $g|_{t=0} = u_0|_\Gamma$ *and if* $p > 3/2$ *also* $g_t|_{t=0} = u_1|_\Gamma$ *in the sense of traces.*

Now we prove optimal regularity for the heat equation and the linearized Westervelt equation with Neumann boundary conditions. Recall that $\lambda_1^N > 0$ denotes the smallest eigenvalue of the negative homogeneous Neumann Laplacian in $L_{p,0}(\Omega)$. Moreover, let $\bar{u} = |\Omega|^{-1} \int_\Omega u \,\mathrm{d}x$ denote the mean of a function $u \colon \Omega \to \mathbb{R}$, whereas $\bar{g} = |\Gamma|^{-1} \int_\Gamma g \, dS$ for $g \colon \Gamma \to \mathbb{R}$.

LEMMA 2.4. *Let* $p \in (1, \infty)\backslash\{3\}$ *and* $\omega \in [0, a\lambda_1^N)$. *Then the inhomogeneous Neumann boundary value problem for the heat equation*

$$\begin{cases} u_t - a\Delta u = f & \text{in } \mathbb{R}_+ \times \Omega, \\ \partial_\nu u = g & \text{on } \mathbb{R}_+ \times \Gamma, \\ u = u_0 & \text{on } \{t = 0\} \times \Omega, \end{cases} \tag{2.11}$$

admits a unique solution of the form $u(t, x) = v(t, x) + w(t)$ *with*

$$v \in \mathrm{e}^{-\omega}\mathbb{H}_{u,0}, \quad \mathbb{H}_{u,0} = W_p^1(\mathbb{R}_+; L_{p,0}(\Omega)) \cap L_p(\mathbb{R}_+; W_p^2(\Omega) \cap L_{p,0}(\Omega)),$$
$$w_t \in \mathrm{e}^{-\omega} L_p(\mathbb{R}_+),$$

if and only if the data satisfy the following conditions:
(i) $f \in \mathrm{e}^{-\omega} L_p(\mathbb{R}_+; L_p(\Omega))$,
(ii) $u_0 \in W_p^{2-2/p}(\Omega)$,
(iii) $g \in \mathrm{e}^{-\omega}\mathbb{H}_\nu$, $\mathbb{H}_\nu = W_p^{1/2-1/2p}(\mathbb{R}_+; L_p(\Gamma)) \cap L_p(\mathbb{R}_+; W_p^{1-1/p}(\Gamma))$,
(iv) $g|_{t=0} = \partial_\nu u_0|_\Gamma$ *in the sense of traces if* $p > 3$.
If in addition $f(t, \cdot)$, u_0, $g(t, \cdot)$ *have mean value zero over* Ω *resp.* Γ *for all* t, *then* $w = 0$.

Proof. We first let $\omega = 0$. By [10, Theorem 8.2], [5, Lemma 4.6] and [14, Theorem 2.4], the Neumann Laplacian in $L_{p,0}(\Omega)$ with domain $\mathcal{D}(\Delta_{N,0}) = \mathcal{D}(\Delta_N) \cap L_{p,0}(\Omega)$ has maximal regularity on $\mathbb{R}_+$. We therefore obtain a unique solution $u_3 \in \mathbb{H}_{u,0}$ of the problem

$$\partial_t u_3 - a\Delta u_3 = f_3 \text{ in } \mathbb{R}_+ \times \Omega, \qquad \partial_\nu u_3 = 0 \text{ on } \mathbb{R}_+ \times \Gamma, \qquad u_3(0) = 0 \text{ in } \Omega$$

for every given $f_3 \in L_p(\mathbb{R}_+; L_{p,0}(\Omega))$. Furthermore, problem (2.11) admits at most one solution. Indeed, let us construct it as $u = u_1 + u_2 + u_3$ where we first solve

$$\partial_t u_1 + \mu u_1 - a\Delta u_1 = 0 \text{ in } \mathbb{R}_+ \times \Omega, \qquad \partial_\nu u_1 = g \text{ on } \mathbb{R}_+ \times \Gamma, \qquad u_1(0) = u_0 \text{ in } \Omega,$$

for some sufficiently large $\mu > 0$ with [11, Theorem 2.1]. Next, we let u_2 solve the ordinary differential equation

$$\partial_t u_2(t) = \bar{f}(t) + \mu\bar{u}_1(t), \qquad u_2(0) = 0,$$

Finally, with $f_3 = f - \bar{f} + \mu(u_1 - \bar{u}_1)$, we obtain u_3 as above. It is easy to check that $v = u_1 + u_3$ and $w = u_2$ satisfy the assertion. The case $\omega > 0$ can be reduced to the previous one by multiplying the functions u, f, g with $t \mapsto \mathrm{e}^{\omega t}$ and using that the spectrum of $-a\Delta_{N,0} + \omega$ is contained in $(0, \infty)$. $\square$

Based on Lemma 2.4, we arrive at our next intermediate result on the way to optimal regularity for the linearized Westervelt equation with Neumann boundary conditions.

LEMMA 2.5. *Let $p \in (1, \infty)\backslash\{3\}$ and $\omega \in (0, b\lambda_1^N)$. Then the inhomogeneous Neumann boundary value problem*

$$\begin{cases} u_{tt} - b\Delta u_t = f & \text{in } \mathbb{R}_+ \times \Omega, \\ \partial_\nu u = g & \text{on } \mathbb{R}_+ \times \Gamma, \\ (u, u_t) = (u_0, u_1) & \text{on } \{t = 0\} \times \Omega, \end{cases} \tag{2.12}$$

admits a unique solution of the form $u(t, x) = v(t, x) + w(t)$ with

$$v \in \mathrm{e}^{-\omega}\mathbb{W}_{u,0}, \quad \mathbb{W}_{u,0} = W_p^2(\mathbb{R}_+; L_{p,0}(\Omega)) \cap W_p^1(\mathbb{R}_+; W_p^2(\Omega) \cap L_{p,0}(\Omega)),$$
$$w \in L_{p,loc}([0, \infty)), \ w_{tt} \in \mathrm{e}^{-\omega}L_p(\mathbb{R}_+),$$

if and only if the data satisfy the following conditions:

- *(i)* $f \in \mathrm{e}^{-\omega}L_p(\mathbb{R}_+ \times \Omega)$,
- *(ii)* $u_0 \in W_p^2(\Omega)$, $u_1 \in W_p^{2-2/p}(\Omega)$,
- *(iii)* $g \in \mathrm{e}^{-\omega}\mathbb{W}_\nu$, $\mathbb{W}_\nu = W_p^{3/2-1/2p}(\mathbb{R}_+; L_p(\Gamma)) \cap W_p^1(\mathbb{R}_+; W_p^{1-1/p}(\Gamma))$,
- *(iv)* $g|_{t=0} = \partial_\nu u_0|_\Gamma$ *and if* $p > 3$ *also* $g_t|_{t=0} = \partial_\nu u_1|_\Gamma$ *in the sense of traces,*
- *(v)* $\int_0^\infty f(t)\, dt = b\Delta u_0 - u_1$.

Moreover, w satisfies $w_{tt}(t) = \bar{f}(t) + b|\Gamma||\Omega|^{-1}\bar{g}_t(t)$ with $w(0) = \bar{u}_0$ and $w_t(0) = \bar{u}_1$.

Proof. We start by proving sufficiency. From the boundedness of $\partial_t : \mathbb{W}_\nu \to \mathbb{H}_\nu$, we infer that $g_t \in \mathrm{e}^{-\omega}\mathbb{H}_\nu$. Therefore, from Lemma 2.4 we obtain that the heat problem

$$\varphi_t - b\Delta\varphi = f \text{ in } \mathbb{R}_+ \times \Omega, \qquad \partial_\nu\varphi = g_t \text{ on } \mathbb{R}_+ \times \Gamma, \qquad \varphi(0) = u_1 \text{ in } \Omega,$$

admits a unique solution of the form $\varphi(t, x) = \varphi_1(t, x) + \varphi_2(t)$ such that $\varphi_1 \in \mathrm{e}^{-\omega}\mathbb{H}_{u,0}$ and $\partial_t\varphi_2 \in \mathrm{e}^{-\omega}L_p(\mathbb{R}_+)$. In particular, since φ_1 has zero mean over Ω, we have $\varphi_2(0) = \bar{\varphi}(0) = \bar{u}_1$ and $\varphi_1(0) = u_1 - \bar{u}_1$. For $x \in \Omega$ and $t \in \mathbb{R}_+$, we define $u(t, x) = v(t, x) + w(t)$, where

$$v(t, x) = -\int_t^\infty \varphi_1(s, x)\, \mathrm{d}s \qquad \text{and} \qquad w(t) = -\int_t^\infty \varphi_2(s)\, \mathrm{d}s.$$

Clearly $u_t = \varphi$; hence, $u_{tt} - b\Delta u_t = f$ in Ω. Integrating the latter with respect to space, multiplying with $|\Omega|^{-1}$ and using the identity $\int_\Omega \Delta u\, \mathrm{d}x = \int_\Gamma \partial_\nu u\, dS$, we conclude that

$$w_{tt}(t) = \bar{f}(t) + b|\Gamma||\Omega|^{-1}\bar{g}_t(t), \quad w(0) = \bar{u}_0, \quad w_t(0) = \bar{u}_1.$$

This implies $v_{tt} - b\Delta v_t = f - \bar{f} - b|\Gamma||\Omega|^{-1}\bar{g}_t$ in Ω. We abbreviate $v(t) = v(t,\cdot)$, $\varphi_1(t) = \varphi_1(t,\cdot)$, etc. and we let $\chi_{\mathbb{R}_-}(t) = 1$ for $t < 0$ and $\chi_{\mathbb{R}_-}(t) = 0$ for $t > 0$. Using that $\omega > 0$, $v_t = \varphi_1$,

$$\mathrm{e}^{\omega t} v(t) = -\int_t^\infty \mathrm{e}^{\omega(t-s)}\mathrm{e}^{\omega s}\varphi_1(s)\,\mathrm{d}s = -(\mathrm{e}^{\omega\cdot}\chi_{\mathbb{R}_-}) * (\mathrm{e}^{\omega\cdot}\varphi_1)(t)$$

together with Young's inequality, we see that $v \in \mathrm{e}^{-\omega}\mathbb{W}_{u,0}$. Moreover, we have

$$\begin{aligned}\partial_\nu v(t)|_\Gamma = \partial_\nu u(t)|_\Gamma &= -\int_t^\infty \partial_\nu\varphi(s)|_\Gamma\,\mathrm{d}s = -\int_t^\infty g_s(s)\,\mathrm{d}s = g(t),\\ v_t(t) = \varphi_1(t) &= -\int_t^\infty \partial_s\varphi_1(s)\,\mathrm{d}s = -\int_t^\infty (f(s) + b\Delta\varphi_1(s) - \partial_s\varphi_2(s))\,\mathrm{d}s\\ &= -\int_t^\infty f(s)\,\mathrm{d}s + b\Delta v(t) - \varphi_2(t),\end{aligned}$$

and $v_t(0) = \varphi_2(0) = u_1 - \bar{u}_1$. Altogether, $b\Delta v(0) = v_t(0) + \bar{u}_1 + \int_0^\infty f(s)\,\mathrm{d}s = b\Delta(u_0 - \bar{u}_0)$ and $\partial_\nu v(0)|_\Gamma = g(0) = \partial_\nu(u_0 - \bar{u}_0)$ which implies that $v(0) = u_0 - \bar{u}_0$ in Ω.

To verify the necessity of (i)–(v), we assume that $u(t,x) = v(t,x) + w(t)$ with $v \in \mathrm{e}^{-\omega}\mathbb{W}_{u,0}$ and $w_{tt} \in \mathrm{e}^{-\omega}L_p(\mathbb{R}_+)$ is a solution of (2.12). It is straightforward to check that $\mathrm{e}^{\omega t} f = \mathrm{e}^{\omega t} v_{tt} + \mathrm{e}^{\omega t} w_{tt} + b\Delta(\mathrm{e}^{\omega t} v_t) \in L_p(\mathbb{R}_+ \times \Omega)$ which proves (i). The desired regularity of the initial values u_0 and u_1 is obtained by using the embedding $W_p^1(J) \hookrightarrow BUC(J)$ and the temporal trace (2.4), respectively. In order to check (iii), we apply the spatial trace (2.2) with $k = l = 1$ to $\mathrm{e}^{\omega t} v \in \mathbb{W}_{u,0}$. Using $W_p^1(J) \hookrightarrow BUC(J)$, (2.3), (2.4), one shows that

$$\partial_\nu u_0|_\Gamma = g|_{t=0} \text{ in } W_p^{1-1/p}(\Gamma) \qquad \text{and} \qquad \partial_\nu u_1|_\Gamma = g_t|_{t=0} \text{ in } W_p^{1-3/p}(\Omega) \text{ for } p > 3$$

hold in the sense of traces and (iv) is satisfied. Finally, integrating $u_{tt}(t) - b\Delta u_t(t) = f(t)$ with respect to time we obtain $u_t(t) - b\Delta u(t) = -\int_t^\infty f(s)\,\mathrm{d}s$ which, for $t = 0$, implies (v). Finally, we can easily deduce from Lemma 2.4 that (2.12) has at most one solution. □

LEMMA 2.6. *Let* $p \in (1,\infty)\backslash\{3\}$ *and* $\omega_0 = \min\{b\lambda_1^N/2, c^2/b\}$. *Then for every* $\omega \in (0,\omega_0)$ *the initial boundary value problem*

$$\begin{cases} u_{tt} - b\Delta u_t - c^2\Delta u = f, & \text{in } \mathbb{R}_+ \times \Omega,\\ \partial_\nu u = 0, & \text{on } \mathbb{R}_+ \times \Gamma,\\ (u, u_t) = (u_0, u_1) & \text{on } \{t = 0\} \times \Omega,\end{cases} \tag{2.13}$$

has a unique solution

$$u \in \mathrm{e}^{-\omega}\mathbb{W}_{u,0}, \qquad \mathbb{W}_{u,0} = W_p^2(\mathbb{R}_+; L_{p,0}(\Omega)) \cap W_p^1(\mathbb{R}_+; W_p^2(\Omega) \cap L_{p,0}(\Omega))$$

if and only if

(i) $f \in e^{-\omega} L_p(\mathbb{R}_+; L_{p,0}(\Omega))$,

(ii) $u_0 \in W_p^2(\Omega) \cap L_{p,0}(\Omega)$ *and* $u_1 \in W_p^{2-2/p}(\Omega) \cap L_{p,0}(\Omega)$ *such that* $\partial_\nu u_0|_\Gamma = 0$ *and if* $p > 3$ *additionally* $\partial_\nu u_1|_\Gamma = 0$.

Proof. Following [26, Section 2], we obtain that for every $\omega \in [0, \omega_0)$ the operator

$$(\partial_t + \tilde{A}, \gamma_t)\colon\ e^{-\omega}(W_p^1(\mathbb{R}_+; \tilde{X}) \cap L_p(\mathbb{R}_+; \mathcal{D}(\tilde{A}))) \\ \to e^{-\omega} L_p(\mathbb{R}_+; \tilde{X}) \times (\tilde{X}, D(\tilde{A}))_{1-1/p,p}$$

is an isomorphism, where $\tilde{X} = \mathcal{D}(\Delta_{N,0}) \times L_{p,0}(\Omega)$ and $\tilde{A}\colon \mathcal{D}(\tilde{A}) \to \tilde{X}$ is given by

$$\tilde{A} = \begin{pmatrix} 0 & -I \\ -c^2\Delta_{N,0} & -b\Delta_{N,0} \end{pmatrix}, \qquad \mathcal{D}(\tilde{A}) = \mathcal{D}(\Delta_{N,0}) \times \mathcal{D}(\Delta_{N,0}). \tag{2.14}$$

It is then easy to check that $(u, u_t) \in e^{-\omega}(W_p^1(\mathbb{R}_+; \tilde{X}) \cap L_p(\mathbb{R}_+; \mathcal{D}(\tilde{A})))$ implies $u \in e^{-\omega}\mathbb{W}_{u,0}$. Clearly, we have $f \in L_p(\mathbb{R}_+; L_{p,0}(\Omega)$, and therefore, (i) holds. Moreover, from $u_0 \in \mathcal{D}(\Delta_N)$ and $u_1 \in (L_{p,0}(\Omega), \mathcal{D}(\Delta_N))_{1-1/p,p}$ we deduce (ii). □

Finally, we arrive at our optimal regularity result for the linearized Westervelt equation with inhomogeneous Neumann boundary conditions.

LEMMA 2.7. *Let* $p \in (1, \infty)\backslash\{3\}$ *and set* $\omega_0 = \min\{b\lambda_1^N/2, c^2/b\}$. *Then for every* $\omega \in (0, \omega_0)$ *the linear initial boundary value problem*

$$\begin{cases} u_{tt} - b\Delta u_t - c^2\Delta u = f, & \text{in } \mathbb{R}_+ \times \Omega, \\ \partial_\nu u = g, & \text{on } \mathbb{R}_+ \times \Gamma, \\ (u, u_t) = (u_0, u_1) & \text{on } \{t = 0\} \times \Omega, \end{cases} \tag{2.15}$$

admits a unique solution of the form $u(t, x) = v(t, x) + w(t)$, *where*

$$v \in e^{-\omega}\mathbb{W}_{u,0}, \quad \mathbb{W}_{u,0} = W_p^2(\mathbb{R}_+; L_{p,0}(\Omega)) \cap W_p^1(\mathbb{R}_+; W_p^2(\Omega) \cap L_{p,0}(\Omega)), \\ w_{tt} \in e^{-\omega} L_p(\mathbb{R}_+)$$

if and only if the data satisfy the following conditions:

(i) $f \in e^{-\omega} L_p(\mathbb{R}_+ \times \Omega)$,

(ii) $u_0 \in W_p^2(\Omega)$, $u_1 \in W_p^{2-2/p}(\Omega)$,

(iii) $g \in e^{-\omega}\mathbb{W}_\nu$, $\mathbb{W}_\nu = W_p^{3/2-1/2p}(\mathbb{R}_+; L_p(\Gamma)) \cap W_p^1(\mathbb{R}_+; W_p^{1-1/p}(\Gamma))$,

(iv) $g|_{t=0} = \partial_\nu u_0|_\Gamma$ *and if* $p > 3$ *additionally* $g_t|_{t=0} = \partial_\nu u_1|_\Gamma$ *in the sense of traces.*

Proof. From Lemma 2.6, we obtain uniqueness. Necessity of (i)–(iv) is shown as in Lemma 2.5. It therefore remains to show sufficiency. Let $\delta > \omega$. From Lemma 2.5, we obtain that

$$\begin{cases} \varphi_{tt} - b\Delta\varphi_t = f - f_\delta & \text{in } \mathbb{R}_+ \times \Omega, \\ \partial_\nu \varphi = g, & \text{on } \mathbb{R}_+ \times \Gamma, \\ (\varphi, \varphi_t) = (u_0, u_1) & \text{on } \{t = 0\} \times \Omega, \end{cases}$$

where $f_\delta = \delta e^{-\delta t}(\int_0^\infty f(s)\,ds + u_1 - b\Delta u_0)$, admits a unique solution $\varphi(x,t) = \varphi_v(x,t) + \varphi_w(t)$ such that $\varphi_v \in e^{-\omega}\mathbb{W}_{u,0}$ and $\partial_t^2 \varphi_w \in e^{-\omega} L_p(\mathbb{R}_+)$. Next, Lemma 2.6 implies that

$$\begin{cases} \theta_{v,tt} - b\Delta\theta_{v,t} - c^2\Delta\theta_v = f_\delta - \bar{f_\delta} + c^2\Delta\varphi_v - c^2\overline{\Delta\varphi_v} & \text{in } \mathbb{R}_+ \times \Omega, \\ \partial_\nu \theta_v = 0, & \text{on } \mathbb{R}_+ \times \Gamma, \\ (\theta_v, \theta_{v,t}) = (0,0) & \text{on } \{t=0\} \times \Omega, \end{cases}$$

has a unique solution $\theta \in e^{-\omega}\mathbb{W}_{u,0}$. Furthermore, we define θ_w as the solution of

$$\theta_{w,tt}(t) = c^2\overline{\Delta\varphi_v}(t) + \bar{f_\delta}(t), \qquad \theta_w(0) = 0, \quad \theta_{w,t}(0) = 0.$$

Then $v = \varphi_v + \theta_v$ and $w = \varphi_w + \theta_w$ satisfy the assertion and we are done. □

REMARK 2.8. If we consider (2.15) on a finite time interval $J = (0,T)$ instead of $\mathbb{R}_+$, we may set $\omega = 0$ and obtain a unique solution $u \in W_p^2(J; L_p(\Omega)) \cap W_p^1(J; W_p^2(\Omega))$ if and only if conditions (i)–(iv) (with $\omega = 0$) hold.

3. The Dirichlet boundary value problem

In this section, we prove global well-posedness and exponential stability for (1.5). First of all, we consider the linearized version of the inhomogeneous Dirichlet boundary value problem and represent it as an abstract evolution equation. We show that this abstract equation admits maximal L_p-regularity and derive an optimal regularity result for the linearized equation associated with (1.5). Then we use the implicit function theorem to construct a solution of the nonlinear problem (1.5). Exponential decay of this solution is an immediate consequence.

3.1. Maximal L_p-regularity for the linearized equation

Suppose $J = (0,T)$ or $J = \mathbb{R}_+$. For $f \in L_p(J \times \Omega)$, we consider the initial boundary value problem

$$\begin{cases} (a\Delta - \partial_t)(u_{tt} - b\Delta u_t - c^2\Delta u) = f & \text{in } J \times \Omega, \\ (u, \Delta u) = (g, h) & \text{on } J \times \Gamma, \\ (u, u_t, u_{tt}) = (u_0, u_1, u_2) & \text{on } \{t=0\} \times \Omega, \end{cases} \tag{3.1}$$

where $u_0, u_1, u_2 \colon \Omega \to \mathbb{R}$ and $g, h \colon J \times \Gamma \to \mathbb{R}$ are the given initial and boundary data, respectively. We say that the data (g, h, u_0, u_1, u_2) are compatible if $u_0|_\Gamma = g|_{t=0}$, $u_1|_\Gamma = g_t|_{t=0}$, $\Delta u_0|_\Gamma = h|_{t=0}$ and, if $p > 3/2$, also $\Delta u_1|_\Gamma = h_t|_{t=0}$, $u_2|_\Gamma = g_{tt}|_{t=0}$.

In order to address the problem of maximal L_p-regularity for the linearized equation, we represent (3.1) with $g = h = 0$ as an abstract Cauchy problem

$$\left(\partial_t + \begin{pmatrix} 0 & -I & 0 \\ -c^2\Delta_D & -b\Delta_D & -I \\ 0 & 0 & -a_D\Delta \end{pmatrix}\right) \begin{pmatrix} u \\ u_t \\ u_{tt} - b\Delta_D u_t - c^2\Delta u_D \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -f \end{pmatrix}.$$

This motivates us to consider the Banach space

$$X^D = \mathcal{D}((\Delta_D)^2) \times \mathcal{D}(\Delta_D) \times L_p(\Omega) \tag{3.2}$$

and the densely defined linear operator $A^D \colon \mathcal{D}(A^D) \to X^D$ given by

$$A^D = \begin{pmatrix} 0 & -I & 0 \\ -c^2\Delta_D & -b\Delta_D & -I \\ 0 & 0 & -a\Delta_D \end{pmatrix},$$
$$\mathcal{D}(A^D) = \mathcal{D}((\Delta_D)^2) \times \mathcal{D}((\Delta_D)^2) \times \mathcal{D}(\Delta_D). \tag{3.3}$$

Therewith, we may write (3.1) as an abstract evolution equation

$$\partial_t v^D + Av^D = F, \qquad v^D(0) = v_0^D$$

if we define

$$v^D = \begin{pmatrix} u \\ u_t \\ u_{tt} - b\Delta_D u_t - c^2\Delta_D u \end{pmatrix},$$
$$v_0^D = \begin{pmatrix} u_0 \\ u_1 \\ u_2 - b\Delta_D u_1 - c^2\Delta_D u_0 \end{pmatrix}, \quad F \begin{pmatrix} 0 \\ 0 \\ -f \end{pmatrix}. \tag{3.4}$$

First of all, we will treat the issue of maximal L_p-regularity of $A^D \colon \mathcal{D}(A^D) \to X^D$ on $\mathbb{R}_+$.

PROPOSITION 3.1. *Let $p \in (1, \infty)$. There is a constant $\mu > 0$ such that $\mu + A^D$ has maximal L_p-regularity on $\mathbb{R}_+$.*

Proof. Let $\alpha > 0$. We decompose A^D, $A^D = A_1^D + A_2^D$, where

$$A_1^D = \begin{pmatrix} \alpha I & -I & 0 \\ 0 & -b\Delta_D & -I \\ 0 & 0 & -a\Delta_D \end{pmatrix} \quad \text{and} \quad A_2^D = \begin{pmatrix} -\alpha I & 0 & 0 \\ -c^2\Delta_D & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

First we show that $A_1^D \colon \mathcal{D}(A^D) \to X^D$ has maximal L_p-regularity. To this end, we consider the Cauchy problem $v_t + A_1^D v = F$, $v_0 = 0$ and show that for each $F \in L_p(\mathbb{R}_+; X^D)$ there exists a unique solution $v \in W_p^1(\mathbb{R}_+; X^D) \cap L_p(\mathbb{R}_+; \mathcal{D}(A^D))$. With $v = (v_1, v_2, v_3)^\top$ and $F = (f_1, f_2, f_3)^\top$, we explicitly have

$$\begin{aligned} \partial_t v_1 + \alpha v_1 - v_2 &= f_1, & v_1(0) &= 0, \\ \partial_t v_2 - b\Delta_D v_2 - v_3 &= f_2, & v_2(0) &= 0, \\ \partial_t v_3 - a\Delta_D v_3 &= f_3, & v_3(0) &= 0. \end{aligned}$$

Since we know from Lemma 2.2 that the homogeneous heat equation admits maximal L_p-regularity, we obtain that for all $f_3 \in L_p(\mathbb{R}_+ \times \Omega)$ there exists a unique solution

$v_3 \in W_p^1(\mathbb{R}_+; L_p(\Omega)) \cap L_p(\mathbb{R}_+; \mathcal{D}(\Delta_D))$. Moreover, as $f_2 + v_3 \in L_p(\mathbb{R}_+; \mathcal{D}(\Delta_D))$, Lemma B.5 implies that there is a unique solution $v_2 \in W_p^1(\mathbb{R}_+; \mathcal{D}(\Delta_D)) \cap L_p(\mathbb{R}_+; \mathcal{D}((\Delta_D)^2))$. Now, note that for $\alpha > 0$ the operator $(\partial_t + \alpha) \colon {}_0W_p^1(\mathbb{R}_+; \mathcal{D}((\Delta_D)^2)) \to L_p(\mathbb{R}_+; \mathcal{D}((\Delta_D)^2))$ is invertible. Since $f_1 + v_2 \in L_p(\mathbb{R}_+; \mathcal{D}((\Delta_D)^2))$, via $v_1(t) = \int_0^t \mathrm{e}^{-\alpha(t-s)}(f_1(s) + v_2(s))\,\mathrm{d}s$ we obtain a unique solution $v_1 \in W_p^1(\mathbb{R}_+; \mathcal{D}((\Delta_D)^2))$. Altogether, we conclude that $A_1 \colon \mathcal{D}(A^D) \to X^D$ admits maximal L_p-regularity.

Moreover, since $A_2^D \colon X^D \to X^D$ is bounded, Proposition 4.3 and Theorem 4.4 in [10] imply that there exists some $\mu > 0$ such that $\mu + A_1^D + A_2^D$ has maximal L_p-regularity. □

To prove maximal regularity for the operator A^D, we need information about its spectrum.

LEMMA 3.2. (cf. [8, Lemma 3.10]) *The spectral bound* $s(-A^D) = \sup\{\mathrm{Re}(\lambda) \colon \lambda \in \sigma(-A^D)\}$ *of* $-A^D$ *is given by* $s(-A^D) = -\omega_0^D$, *where* $\omega_0^D = \min\{a\lambda_0^D, b\lambda_0^D/2, c^2/b\}$. *In particular, if* $\mathrm{Re}(\lambda) < \omega_0^D$, *then* $\lambda \in \rho(A^D)$.

THEOREM 3.3. *Let* $p \in (1,\infty)$ *and* $\omega \in [0, \omega_0)$ *where* $\omega_0^D = \min\{a\lambda_0^D, b\lambda_0^D/2, c^2/b\}$. *Then* $A^D \colon \mathcal{D}(A^D) \to X^D$ *has maximal* L_p*-regularity on* $\mathbb{R}_+$ *in the sense that*

$$
\begin{aligned}
(\partial_t + A^D, \gamma_t) \colon \mathrm{e}^{-\omega}(W_p^1(\mathbb{R}_+; X^D) \cap L_p(\mathbb{R}_+; \mathcal{D}(A^D))) \\
\to \mathrm{e}^{-\omega} L_p(\mathbb{R}_+; X^D) \times (X^D, \mathcal{D}(A^D))_{1-1/p,p}
\end{aligned}
$$

is an isomorphism.

Proof. We follow the proof of Theorem 2.5 in [26]. From Proposition 3.1, we know $\mu + A^D$ admits maximal regularity on $\mathbb{R}_+$ for some $\mu > 0$. Multiplying $v_t^D + A^D v^D = F$ by $\mathrm{e}^{-\mu t}$ shows that A^D has maximal L_p-regularity on bounded intervals $J = (0, T)$. Lemma 3.2 tells us that spectral bound $s(-A^D) = -\omega_0$ of $-A^D$ is strictly negative. Hence, $s(-A^D + \omega) = \omega - \omega_0^D < 0$ as long as $\omega \in [0, \omega_0^D)$. From [14, Theorem 2.4], we deduce that $A^D - \omega$ admits maximal L_p-regularity on $\mathbb{R}_+$ for every $\omega \in [0, \omega_0^D)$, that is,

$$
\begin{aligned}
(\partial_t + A^D - \omega, \gamma_t) \colon\ W_p^1(\mathbb{R}_+; X^D) \cap L_p(\mathbb{R}_+; \mathcal{D}(A^D)) \\
\to L_p(\mathbb{R}_+; X^D) \times (X^D, \mathcal{D}(A^D))_{1-1/p,p}
\end{aligned}
$$

is an isomorphism. Now Lemma 2.1 implies the result. □

COROLLARY 3.4. *Let* $p \in (1,\infty)\backslash\{3/2\}$ *and define* $\omega_0^D = \min\{a\lambda_0^D, b\lambda_0^D/2, c^2/b\}$. *Then for every* $\omega \in (0, \omega_0)$ *the linear initial boundary value problem* (3.1) *on* $J = \mathbb{R}_+$ *with* $g = h = 0$ *admits maximal* L_p*-regularity in the sense that there exists a unique solution*

$$
u \in \mathrm{e}^{-\omega}\mathbb{E}_u, \quad \mathbb{E}_u = W_p^3(\mathbb{R}_+; L_p(\Omega)) \cap W_p^1(\mathbb{R}_+; W_p^4(\Omega)),
$$

if and only if

(i) $f \in \mathrm{e}^{-\omega} L_p(\mathbb{R}_+ \times \Omega)$,

(ii) $u_0 \in W_p^4(\Omega)$, $u_1 \in W_p^{4-2/p}(\Omega)$, $u_2 \in W_p^{2-2/p}(\Omega)$ *with compatibility of the data.*

Proof. Based on the choices of X^D and $\mathcal{D}(A^D)$ in (3.2) and (3.3), it is straightforward to check that the condition $v^D \in \mathrm{e}^{-\omega}(W_p^1(\mathbb{R}_+; X^D) \cap L_p(\mathbb{R}_+; \mathcal{D}(A^D)))$ where v^D is given by (3.4) implies $u \in \mathrm{e}^{-\omega}(\mathbb{E}_u \cap W_p^2(\mathbb{R}_+; W_p^2(\Omega)))$. Since the mixed derivative embedding gives us $\mathbb{E}_u \hookrightarrow W_p^2(\mathbb{R}_+; W_p^2(\Omega))$, we arrive at $u \in \mathrm{e}^{-\omega}\mathbb{E}_u$. Furthermore, $F \in L_p(\mathbb{R}_+; X^D)$ leads to (i). Next, we determine $(X, \mathcal{D}(A))_{1-1/p,p}$. It is trivial that $(\mathcal{D}((\Delta_D)^2), \mathcal{D}((\Delta_D)^2))_{1-1/p,p} = \mathcal{D}((\Delta_D)^2)$; that is, we have $u_0 \in W_p^4(\Omega)$ with $u|_\Gamma = \Delta u|_\Gamma = 0$. Since for $p \in (1,\infty)$ we have $2/p \in \mathbb{R}\backslash\mathbb{N}$ unless $p = 2$, (2.1) gives us

$$(W_p^2(\Omega), W_p^4(\Omega))_{1-1/p,p} = W_p^{4-2/p}(\Omega),$$
$$(L_p(\Omega), W_p^2(\Omega))_{1-1/p,p} = W_p^{2-2/p}(\Omega).$$

Moreover, interpolation with boundary conditions as in [3, Section 4.9] yields $u_1|_\Gamma = \Delta u_1|_\Gamma = u_2 - b\Delta_D u_1 - c^2\Delta_D u_0|_\Gamma = 0$ and (ii) follows. □

We now arrive at the final result for this section and prove optimal regularity for the linear initial boundary value problem (3.1).

PROPOSITION 3.5. *Let $p \in (1,\infty)\backslash\{3/2\}$ and define $\omega_0^D = \min\{a\lambda_0^D, b\lambda_0^D/2, c^2/b\}$. Then for every $\omega \in (0, \omega_0)$ the linear initial boundary value problem (3.1) on $J = \mathbb{R}_+$ has a unique solution*

$$u \in \mathrm{e}^{-\omega}\mathbb{E}_u, \quad \mathbb{E}_u = W_p^3(\mathbb{R}_+; L_p(\Omega)) \cap W_p^1(\mathbb{R}_+; W_p^4(\Omega)),$$

if and only if the data satisfy the conditions

(i) $f \in \mathrm{e}^{-\omega} L_p(\mathbb{R}_+ \times \Omega)$,

(ii) $u_0 \in W_p^4(\Omega)$, $u_1 \in W_p^{4-2/p}(\Omega)$, $u_2 \in W_p^{2-2/p}(\Omega)$,

(iii) $g \in \mathrm{e}^{-\omega}\mathbb{F}_{g,\Gamma}$, $\mathbb{F}_{g,\Gamma} = W_p^{3-1/2p}(\mathbb{R}_+; L_p(\Gamma)) \cap W_p^1(\mathbb{R}_+; W_p^{4-1/p}(\Gamma))$,
$h \in \mathrm{e}^{-\omega}\mathbb{F}_{h,\Gamma}$, $\mathbb{F}_{h,\Gamma} = W_p^{2-1/2p}(\mathbb{R}_+; L_p(\Gamma)) \cap W_p^1(\mathbb{R}_+; W_p^{2-1/p}(\Gamma))$,

(iv) (g, h, u_0, u_1, u_2) *are compatible.*

Moreover, the solution fulfills the estimate

$$\|u\|_{\mathrm{e}^{-\omega}\mathbb{E}_u} \lesssim \|f\|_{\mathrm{e}^{-\omega}L_p} + \|g\|_{\mathrm{e}^{-\omega}\mathbb{F}_{g,\Gamma}} + \|h\|_{\mathrm{e}^{-\omega}\mathbb{F}_{h,\Gamma}} + \|u_0\|_{W_p^4} + \|u_1\|_{W_p^{4-2/p}} + \|u_2\|_{W_p^{2-2/p}}.$$

Proof. It is not difficult to check that (i)–(iv) are necessary for the regularity of the solution, by using Sobolev's embedding $W_p^1(J) \hookrightarrow BUC(J)$, the spatial trace theorem (2.3), the temporal trace theorem (2.4) and the mixed derivative embeddings (2.5). In particular, the compatibility conditions are understood in the following sense:

$$u_0|_\Gamma = g|_{t=0} \text{ in } W_p^{4-1/p}(\Gamma), \qquad \Delta u_0|_\Gamma = h|_{t=0} \text{ in } W_p^{2-1/p}(\Gamma),$$
$$u_1|_\Gamma = g_t|_{t=0} \text{ in } B_{p,p}^{4-3/p}(\Gamma), \qquad \Delta u_1|_\Gamma = h_t|_{t=0} \text{ in } B_{p,p}^{2-3/p}(\Gamma) \text{ if } p > 3/2,$$
$$u_2|_\Gamma = g_{tt}|_{t=0} \text{ in } B_{p,p}^{2-3/p}(\Gamma) \text{ if } p > 3/2.$$

It remains to show that conditions (i)–(iv) imply the existence of a unique solution $u \in \mathrm{e}^{-\omega}\mathbb{E}_u$. As we are dealing with a linear partial differential equation with constant coefficients, we may interchange the order of differentiation and consider the subproblems

$$\begin{cases} w_{tt} - b\Delta w_t - c^2\Delta w = f & \text{in } \mathbb{R}_+ \times \Omega, \\ w = ah - g_t & \text{on } \mathbb{R}_+ \times \Gamma, \\ (w, w_t) = (a\Delta u_0 - u_1, a\Delta u_1 - u_2) & \text{on } \{t=0\} \times \Omega, \end{cases} \tag{3.5}$$

and

$$\begin{cases} a\Delta u - u_t = w & \text{in } \mathbb{R}_+ \times \Omega, \\ u = g & \text{on } \mathbb{R}_+ \times \Gamma, \\ u = u_0 & \text{on } \{t=0\} \times \Omega. \end{cases} \tag{3.6}$$

From condition (ii), we obtain $a\Delta u_0 - u_1 \in W_p^2(\Omega)$ and $a\Delta u_1 - u_2 \in W_p^{2-2/p}(\Omega)$. Furthermore, (iii) implies $ah - g_t \in \mathrm{e}^{-\omega}\mathbb{W}_\Gamma$. By Lemma 2.3, problem (3.5) admits a unique solution $w \in \mathrm{e}^{-\omega}\mathbb{W}_u$. Now Corollary B.4 with $l = 1$ and $k = 2$ shows that (3.6) has a solution $u \in \mathrm{e}^{-\omega}\mathbb{E}_u$. This yields sufficiency and uniqueness follows from Corollary 3.4. □

3.2. Global well-posedness and exponential stability

Based on Proposition 3.5, we now show that there exists a unique global solution of the nonlinear initial boundary value problem (1.5) which depends continuously (in fact, even analytically) on the (sufficiently small) initial and boundary data. Moreover, we prove that the equilibrium $u = 0$ is exponentially stable. For a detailed treatment of analytic mappings in Banach spaces and the analytic version of the implicit function theorem, we refer to [9, Section 15.1].

THEOREM 3.6. (Global well-posedness: the Dirichlet case) *Let* $p > \max\{n/4 + 1/2, n/3\}$, $p \neq 3/2$ *and define* $\omega_0^D = \min\{a\lambda_0^D, b\lambda_0^D/2, c^2/b\}$. *Suppose*

$$\begin{aligned} &u_0 \in W_p^4(\Omega), \quad u_1 \in W_p^{4-2/p}(\Omega), \quad u_2 \in W_p^{2-2/p}(\Omega) \\ &g \in \mathrm{e}^{-\omega}\mathbb{F}_{g,\Gamma},\ \mathbb{F}_{g,\Gamma} = W_p^{3-1/2p}(\mathbb{R}_+; L_p(\Gamma)) \cap W_p^1(\mathbb{R}_+; W_p^{4-1/p}(\Gamma)), \\ &h \in \mathrm{e}^{-\omega}\mathbb{F}_{h,\Gamma},\ \mathbb{F}_{h,\Gamma} = W_p^{2-1/2p}(\mathbb{R}_+; L_p(\Gamma)) \cap W_p^1(\mathbb{R}_+; W_p^{2-1/p}(\Gamma)) \end{aligned} \tag{3.7}$$

and assume that the data (g, h, u_0, u_1, u_2) *are compatible.*

Then for every $\omega \in (0, \omega_0)$ *there exists some* $\rho > 0$ *such that if*

$$\|g\|_{\mathrm{e}^{-\omega}\mathbb{F}_{g,\Gamma}} + \|h\|_{\mathrm{e}^{-\omega}\mathbb{F}_{h,\Gamma}} + \|u_0\|_{W_p^4} + \|u_1\|_{W_p^{4-2/p}} + \|u_2\|_{W_p^{2-2/p}} < \rho, \tag{3.8}$$

the nonlinear initial boundary value problem (1.5) *admits a unique solution*

$$u \in \mathrm{e}^{-\omega}\mathbb{E}_u, \qquad \mathbb{E}_u = W_p^3(\mathbb{R}_+; L_p(\Omega)) \cap W_p^1(\mathbb{R}_+; W_p^4(\Omega)) \tag{3.9}$$

which depends analytically on the data (3.7) *with respect to the corresponding topologies. Moreover, conditions* (3.7) *are necessary for the regularity of the solution* (3.9).

Proof. Employing the results on the linearized problem (3.1) from Sect. 3.1, we will now construct a solution of the nonlinear initial boundary value problem (1.5) which we linearize at $u = 0$. Hence, the solution will be of the form $u = u_\star + u_\bullet$, where $u_\star$ solves the linearized problem (3.1) for the data $(f = 0, g, h, u_0, u_1, u_2)$ and $u_\bullet$ satisfies homogeneous boundary and initial conditions. We will find the (small) deviation $u_\bullet$ from $u_\star$ by application of the implicit function theorem to the map

$$\begin{aligned} G\colon \mathrm{e}^{-\omega}\mathbb{E}_{u,h} \times \mathrm{e}^{-\omega}\mathbb{E}_u &\to \mathrm{e}^{-\omega}L_p(\mathbb{R}_+ \times \Omega), \\ (u_\bullet, u_\star) &\mapsto D(\partial_t, \Delta)u_\bullet - (k((u_\bullet + u_\star)_t)^2 - s|\nabla(u_\bullet + u_\star)|^2)_{tt} \end{aligned} \tag{3.10}$$

where the differential expression $D(\partial_t, \Delta)$ is given by $D(\partial_t, \Delta) = (a\Delta - \partial_t)(\partial_t^2 - b\Delta\partial_t - c^2\Delta)$ and $\mathbb{E}_{u,h} = \{u \in \mathbb{E}_u \colon u(0) = u_t(0) = u_{tt}(0) = 0,\ u|_\Gamma = \Delta u|_\Gamma = 0\}$. Explicitly, we have $D(\partial_t, \Delta)u_\bullet = -u_{\bullet,ttt} + (a+b)\Delta u_{\bullet,tt} + c^2\Delta u_{\bullet,t} - ab\Delta^2 u_{\bullet,t} - ac^2\Delta^2 u_\bullet$.

Step 1: The implicit function theorem applies. First of all, we will now verify the assumptions of the analytic version of the implicit function theorem.

Step 1(a): G is analytic. The mixed derivative embeddings (2.5) imply that the linear map $u_\bullet \mapsto D(\partial_t, \Delta)u_\bullet \colon \mathrm{e}^{-\omega}\mathbb{E}_{u,h} \to \mathrm{e}^{-\omega}L_p(\mathbb{R}_+ \times \Omega)$ is bounded and therefore analytic.

For the remaining bilinear operators, we employ the pointwise multiplication estimates $\|fg\|_p \le \|f\|_p\|g\|_\infty$ and $\|fg\|_p \le \|f\|_{2p}\|g\|_{2p}$ and suitable embeddings as in the proof of Lemma 6 in [27]. In order to arrive at the $\|\cdot\|_\infty$-norm, we infer from the mixed derivative embeddings and Sobolev's embedding that the embedding

$$\begin{aligned} \mathbb{E}_u &\hookrightarrow W_p^1(\mathbb{R}_+; W_p^4(\Omega)) \cap W_p^2(\mathbb{R}_+ \times \Omega) \\ &\hookrightarrow H_p^{1+1/p+\varepsilon}(\mathbb{R}_+; H_p^{4-2/p-2\varepsilon}(\Omega)) \hookrightarrow BUC^1(\mathbb{R}_+; BUC(\Omega)), \end{aligned}$$

is valid if $\varepsilon \in (0, 1 - 1/p)$ and $4 - 2/p - 2\varepsilon - n/p > 0$. Such a number ε exists if $p > n/4 + 1/2$. Sobolev's embedding also yields the embedding

$$\mathbb{E}_u \hookrightarrow W_p^1(\mathbb{R}_+; W_p^4(\Omega)) \hookrightarrow BUC(\mathbb{R}_+; W_p^4(\Omega)) \hookrightarrow BUC(\mathbb{R}_+; BUC^1(\Omega))$$

if $p > n/3$. In order to arrive at the $\|\cdot\|_{2p}$-norm, we conclude that

$$\begin{aligned}
\mathbb{E}_u &\hookrightarrow W_p^2(\mathbb{R}_+; W_p^2(\Omega)) \cap W_p^3(\mathbb{R}_+; L_p(\Omega)) \\
&\hookrightarrow H_p^{2+\Theta-\varepsilon/2}(\mathbb{R}_+; H_p^{2-2\Theta+\varepsilon}(\Omega)) && \text{for } \Theta - \tfrac{\varepsilon}{2} \in [0,1] \text{ and } \varepsilon > 0, \\
&\hookrightarrow W_p^{2+\Theta-\varepsilon}(\mathbb{R}_+; W_p^{2-2\Theta}(\Omega)) && \text{for } \varepsilon > 0, \\
&\hookrightarrow W_{2p}^2(\mathbb{R}_+; W_p^{2-2\Theta}(\Omega)) && \text{for } \Theta \geq \tfrac{1}{2p} + \varepsilon, \\
&\hookrightarrow W_{2p}^2(\mathbb{R}_+; L_{2p}(\Omega)) && \text{for } \Theta \leq 1 - \tfrac{n}{4p},
\end{aligned}$$

provided that $\varepsilon > 0$ is sufficiently small and $p > n/4 + 1/2$. Similarly, we obtain

$$\begin{aligned}
\mathbb{E}_u &\hookrightarrow W_p^1(\mathbb{R}_+; W_p^4(\Omega)) \cap W_p^2(\mathbb{R}_+ \times \Omega) \\
&\hookrightarrow H_p^{1+1/2p+\varepsilon}(\mathbb{R}_+; H_p^{4-1/p-2\varepsilon}(\Omega)) \\
&\hookrightarrow W_{2p}^1(\mathbb{R}_+; H_p^{4-1/p-2\varepsilon}(\Omega)) \hookrightarrow W_{2p}^1(\mathbb{R}_+ \times \Omega),
\end{aligned}$$

for some $\varepsilon > 0$, provided that $p > n/6 + 1/3$. Furthermore, since $\mathrm{e}^{\omega t} \leq \mathrm{e}^{2\omega t}$ for $\omega \geq 0$, we observe that $\mathrm{e}^{-2\omega} L_p(\mathbb{R}_+ \times \Omega) \hookrightarrow \mathrm{e}^{-\omega} L_p(\mathbb{R}_+ \times \Omega)$.

Prepared like that, we estimate

$$\begin{aligned}
\|f_t g_t\|_{L_p} &\leq \|f_t\|_{L_{2p}} \|f_t\|_{L_{2p}} \lesssim \|f\|_{\mathbb{E}_u} \|g\|_{\mathbb{E}_u}, \\
\|(f_t g_t)_t\|_{L_p} &\leq \|f_{tt}\|_{L_p} \|g_t\|_{L_\infty} + \|f_t\|_{L_\infty} \|g_{tt}\|_{L_p} \lesssim \|f\|_{\mathbb{E}_u} \|g\|_{\mathbb{E}_u}, \\
\|(f_t g_t)_{tt}\|_{L_p} &\leq \|f_{ttt}\|_{L_p} \|g_t\|_{L_\infty} + 2\|f_{tt}\|_{L_{2p}} \|g_{tt}\|_{L_{2p}} + \|f_t\|_{L_\infty} \|g_{ttt}\|_{L_p} \\
&\lesssim \|f\|_{\mathbb{E}_u} \|g\|_{\mathbb{E}_u},
\end{aligned}$$

and conclude that $(f, g) \mapsto f_t g_t \colon \mathbb{E}_u \times \mathbb{E}_u \to W_p^2(\mathbb{R}_+; L_p(\Omega))$ is bilinear and bounded, thus analytic. Setting $w = u_\bullet + u_\star$ in

$$\mathrm{e}^{2\omega t}((w_t)^2)_{tt} = \tfrac{1}{2}((\mathrm{e}^{\omega t} w_t)^2)_{tt} - 3\omega((\mathrm{e}^{\omega t} w_t)^2)_t + 6\omega^2(\mathrm{e}^{\omega t} w_t)^2 \tag{3.11}$$

and choosing $f = \mathrm{e}^{\omega t} u_\bullet$ and $g = \mathrm{e}^{\omega t} u_\star$ proves that

$$(u_\bullet, u_\star) \mapsto (((u_\bullet + u_\star)_t)^2)_{tt} \colon \mathrm{e}^{-\omega}\mathbb{E}_{u,h} \times \mathrm{e}^{-\omega}\mathbb{E}_u \to \mathrm{e}^{-\omega} L_p(\mathbb{R}_+ \times \Omega) \tag{3.12}$$

is analytic.

It remains to show that $(u_\bullet, u_\star) \mapsto (|\nabla(u_\bullet + u_\star)|^2)_{tt}$ is analytic. To this end, we estimate

$$\begin{aligned}
\|\nabla f \cdot \nabla g\|_{L_p} &\leq \|\nabla f\|_{L_{2p}} \|g\|_{L_{2p}} \lesssim \|f\|_{\mathbb{E}_u} \|g\|_{\mathbb{E}_u}, \\
\|(\nabla f \cdot \nabla g)_t\|_{L_p} &\leq \|(\nabla f)_t\|_{L_{2p}} \|\nabla g\|_{L_{2p}} + \|\nabla f\|_{L_{2p}} \|(\nabla g)_t\|_{L_{2p}} \lesssim \|f\|_{\mathbb{E}_u} \|g\|_{\mathbb{E}_u}, \\
\|(\nabla f \cdot \nabla g)_{tt}\|_{L_p} &\leq \|(\nabla f)_{tt}\|_{L_p} \|\nabla g\|_{L_\infty} + 2\|(\nabla f)_t\|_{L_{2p}} \|(\nabla g)_t\|_{L_{2p}} \\
&\quad + \|\nabla f\|_{L_\infty} \|(\nabla g)_{tt}\|_{L_p} \\
&\lesssim \|f\|_{\mathbb{E}_u} \|g\|_{\mathbb{E}_u},
\end{aligned}$$

and conclude that $(f, g) \mapsto \nabla f \cdot \nabla g \colon \mathbb{E}_u \times \mathbb{E}_u \to W_p^2(\mathbb{R}_+; L_p(\Omega))$ is analytic. Moreover,

$$e^{2\omega t}((\nabla w)^2)_{tt} = ((e^{\omega t}\nabla w)^2)_{tt} - 4\omega((e^{\omega t}\nabla w)^2)_t + 4\omega^2(e^{\omega t}\nabla w)^2.$$

By setting $w = u_\bullet + u_\star$, $f = e^{\omega t}u_\bullet$ and $g = e^{\omega t}u_\star$ we are done.

Step 1(b): $D_{u_\bullet}G(0,0)\colon e^{-\omega}L_p(\mathbb{R}_+ \times \Omega) \to e^{-\omega}\mathbb{E}_{u,h}$ is an isomorphism. The Fréchet derivative of G with respect to $u_\bullet$ at $(0, 0)$ is given by $D_{u_\bullet}G(0,0)[\overline{u}] = (a\Delta - \partial_t)(\overline{u}_{tt} - c^2\Delta\overline{u} - b\Delta\overline{u}_t)$. The map $D_{u_\bullet}G(0,0)\colon e^{-\omega}L_p(\mathbb{R}_+ \times \Omega) \to e^{-\omega}\mathbb{E}_{u,h}$ is an isomorphism since, according to Corollary 3.4, for every $\overline{f} \in e^{-\omega}L_p(\mathbb{R}_+ \times \Omega)$ the equation $(a\Delta - \partial_t)(\overline{u}_{tt} - c^2\Delta\overline{u} - b\Delta\overline{u}_t) = \overline{f}$ admits a unique solution $\overline{u} \in e^{-\omega}\mathbb{E}_{u,h}$.

Step 2: Construction of the solution. On the strength of the implicit function theorem, there exists a ball $B_\rho(0) \subset e^{-\omega}\mathbb{E}_u$ with sufficiently small radius $\rho > 0$ and an analytic map $\varphi\colon B_\rho(0) \subset e^{-\omega}\mathbb{E}_u \to e^{-\omega}\mathbb{E}_{u,h}$, $u_\star \mapsto u_\bullet = \varphi(u_\star)$ satisfying $\varphi(0) = 0$ and $G(\varphi(u_\star), u_\star) = 0$ for all $u_\star \in B_\rho(0)$. Hence, whenever $u_\star$ satisfies the boundary conditions $u_\star|_\Gamma = g$, $\Delta u_\star|_\Gamma = h$ and initial conditions $u_\star|_{t=0} = u_0$, $u_{\star,t}|_{t=0} = u_1$, $u_{\star,tt}|_{t=0} = u_2$ which is the case if we define $u_\star \in e^{-\omega}\mathbb{E}_u$ to be the unique solution of (3.1) with $(f = 0, u_0, u_1, u_2, g, h)$, then $u_\bullet + u_\star = \varphi(u_\star) + u_\star$ solves (1.5).

Step 3: Dependence of the solution on the data. It remains to show that the solution $u \in e^{-\omega}\mathbb{E}_u$ depends analytically on (g, h, u_0, u_1, u_2). To this end, we define the spaces

$$\begin{aligned}\overline{\mathbb{D}} &= e^{-\omega}\mathbb{E}_g \times e^{-\omega}\mathbb{E}_h \times W_p^4(\Omega) \times W_p^{4-2/p}(\Omega) \times W_p^{2-2/p}(\Omega),\\ \mathbb{D} &= \{(g, h, u_0, u_1, u_2) \in \overline{\mathbb{D}}\colon u_0|_\Gamma = g|_{t=0},\, u_1|_\Gamma = g_t|_{t=0},\, u_2|_\Gamma = g_{tt}|_{t=0} \text{ if } p > 3/2,\\ &\qquad \Delta u_0|_\Gamma = h|_{t=0},\, \Delta u_1|_\Gamma = h_t|_{t=0} \text{ if } p > 3/2\}.\end{aligned}$$

From Proposition 3.5 with $f = 0$, we obtain that $u_\star$ depends linearly and continuously and thus analytically on $(g, h, u_0, u_1, u_2) \in \mathbb{D}$. Moreover, $u_\star \mapsto u_\bullet = \varphi(u_\star)$ is analytic on $B_\rho(0)$, and therefore, $u_\bullet \in e^{-\omega}\mathbb{E}_{u,h}$ depends analytically on the data $(g, h, u_0, u_1, u_2) \in \mathbb{D}$. Altogether, $u = u_\bullet + u_\star$ enjoys the same property which concludes the proof. $\square$

An immediate consequence of Theorem 3.6 is that the solution decays to zero exponentially.

COROLLARY 3.7. (Exponential stability: the Dirichlet case) *Under the same assumptions as in Theorem 3.6, the solution u decays exponentially fast to zero as $t \to \infty$, in the sense that*

$$\|u(t)\|_{W_p^4} + \|u_t(t)\|_{W_p^{4-2/p}} + \|u_{tt}\|_{W_p^{2-2/p}} \le C e^{-\omega t}, \qquad t \ge 0,$$

for some $C \ge 0$ depending on the boundary and initial data g, h, u_0, u_1 and u_2.

Proof. We have $u \in e^{-\omega}W_p^1(\mathbb{R}_+; W_p^4(\Omega)) \hookrightarrow e^{-\omega}BUC(\mathbb{R}_+; W_p^4(\Omega))$, hence

$$u \in BUC(\mathbb{R}_+, W_p^4(\Omega)), \quad \|u(t)\|_{W_p^4} \le C_1 e^{-\omega t} \text{ with } C_1 = \|e^{\omega\cdot}u\|_{BUC(\mathbb{R}_+; W_p^4)}.$$

Furthermore, $\nabla_x^j u_t \in \mathbb{H}(\mathbb{R}_+) \hookrightarrow BUC(\mathbb{R}_+; W_p^{2-2/p}(\Omega))$ for $j \in \{0, 1, 2\}$. Therefore, we obtain

$$u_t \in BUC(\mathbb{R}_+, W_p^{4-2/p}(\Omega)), \quad \|u_t(t)\|_{W_p^{4-2/p}} \leq C_2 \, \mathrm{e}^{-\omega t}$$

with $C_2 = \|\mathrm{e}^{\omega\cdot} u_t\|_{BUC(\mathbb{R}_+; W_p^{4-2/p})}$. Finally, from $u_{tt} \in \mathbb{H}(\mathbb{R}_+)$ we deduce that

$$u_{tt} \in BUC(\mathbb{R}_+, W_p^{2-2/p}(\Omega)), \quad \|u_{tt}(t)\|_{W_p^{2-2/p}} \leq C_3 \, \mathrm{e}^{-\omega t}$$

with $C_3 = \|\mathrm{e}^{\omega\cdot} u_{tt}\|_{BUC(\mathbb{R}_+; W_p^{2-2/p})}$ and the claim follows. □

4. The Neumann boundary value problem

In this section, we treat the Neumann boundary value problem (1.6). We proceed analogously to the Dirichlet case; that is, we first consider the linearized equation and then construct a solution of the nonlinear problem (1.6) by means of the implicit function theorem.

Note that, in the Dirichlet case, the fact that the operator $-A^D : \mathcal{D}(A^D) \to X^D$ defined by (3.3) has a strictly negative spectral bound (Lemma 3.2) was crucial in order to show that the linearized equation (3.1) has maximal regularity on $\mathbb{R}_+$, see the proof of Theorem 3.3.

In the Neumann case, due to the zero eigenvalue of $-\Delta_N : \mathcal{D}(\Delta_N) \to L_p(\Omega)$ with $\mathcal{D}(\Delta_N) = \{u \in W_p^2(\Omega) : \partial_\nu u = 0 \text{ on } \Gamma\}$, we cannot expect to obtain maximal regularity on $\mathbb{R}_+$. For this reason, we consider $-\Delta_{N,0} : \mathcal{D}(\Delta_{N,0}) \to L_{p,0}(\Omega)$, where $\mathcal{D}(\Delta_{N,0}) = \mathcal{D}(\Delta_N) \cap L_{p,0}(\Omega)$. The spectrum of $-\Delta_{N,0}$ is contained in $(0, \infty)$; therefore, we can prove maximal regularity of the homogeneous linear Neumann boundary problem on $\mathbb{R}_+$ analogously to the Dirichlet case.

However, if we restrict ourselves to finite time intervals $J = (0, T)$, then we do not necessarily need to use $-\Delta_{N,0}$. In case of finite time intervals, we use $-\Delta_N$. As a consequence, we will prove global well-posedness of (1.6) only if the data u_0, u_1, u_2 and g, h have zero mean, whereas local well-posedness holds also for data with nonzero mean.

4.1. Maximal L_p-regularity for the linearized equation

As in Sect. 3.1, let $J = (0, T)$ or $J = \mathbb{R}_+$ and assume $p \in (1, \infty)$. Here, for $f \in L_p(J \times \Omega)$ we consider

$$\begin{cases} (a\Delta - \partial_t)(u_{tt} - b\Delta u_t - c^2 \Delta u) = f & \text{in } J \times \Omega, \\ (\partial_\nu u, \partial_\nu \Delta u) = (g, h) & \text{on } J \times \Gamma, \\ (u, u_t, u_{tt}) = (u_0, u_1, u_2) & \text{on } \{t = 0\} \times \Omega, \end{cases} \tag{4.1}$$

where $u_0, u_1, u_2 : \Omega \to \mathbb{R}$ and $g, h : J \times \Gamma \to \mathbb{R}$ are the given initial and boundary data, respectively. We say that the data (g, h, u_0, u_1, u_2) are compatible, if $\partial_\nu u_0|_\Gamma =$

$g|_{t=0}$, $\partial_\nu \Delta u_0|_\Gamma = h|_{t=0}$, $\partial_\nu u_1|_\Gamma = g_t|_{t=0}$ and, if $p > 3$, also $\partial_\nu \Delta u_1|_\Gamma = h_t|_{t=0}$, $\partial_\nu u_2|_\Gamma = g_{tt}|_{t=0}$ in the sense of traces.

Analogously to the Dirichlet case, we first represent (4.1) with $g = h = 0$ as an abstract evolution equation of the form

$$\partial_t v^N + A^N v^N = F, \qquad v^N(0) = v_0^N \tag{4.2}$$

by setting

$$v^N = \begin{pmatrix} u \\ u_t \\ u_{tt} - c^2\Delta_N u - b\Delta_N u_t \end{pmatrix}, \quad v_0^N = \begin{pmatrix} u_0 \\ u_1 \\ u_2 - c^2\Delta_N u_0 - b\Delta_N u_1 \end{pmatrix}, \quad F = \begin{pmatrix} 0 \\ 0 \\ -f \end{pmatrix},$$

introducing the Banach space $X^N = \mathcal{D}((\Delta_N)^2) \times \mathcal{D}(\Delta_N) \times L_p(\Omega)$ and defining the coefficient operator $A^N : \mathcal{D}(A^N) \to X^N$ via

$$\begin{aligned} A^N &= \begin{pmatrix} 0 & -I & 0 \\ -c^2\Delta_N & -b\Delta_N & -I \\ 0 & 0 & -a\Delta_N \end{pmatrix}, \\ \mathcal{D}(A^N) &= \mathcal{D}((\Delta_N)^2) \times \mathcal{D}((\Delta_N)^2) \times \mathcal{D}(\Delta_N). \end{aligned} \tag{4.3}$$

On the one hand, we will show maximal L_p-regularity of A^N on finite time intervals. On the other hand, we shall use the realization $-\Delta_{N,0}$ of the homogeneous Neumann Laplacian. Therefore, we introduce the Banach space $X^{N,0} = \mathcal{D}((\Delta_{N,0})^2) \times \mathcal{D}(\Delta_{N,0}) \times L_{p,0}(\Omega)$ and the densely defined operator $A^{N,0} : \mathcal{D}(A^{N,0}) \to X^{N,0}$, where Δ_N has to be replaced by $\Delta_{N,0}$ in (4.3).

PROPOSITION 4.1. *Let $p \in (1, \infty)$. There exists some $\nu > 0$ such that the operators $\nu + A^N$ and $\nu + A^{N,0}$ admit maximal regularity on $\mathbb{R}_+$.*

Proof. The result can be proved similarly to Proposition 3.1. For some $\alpha > 0$, consider

$$A_1^N = \begin{pmatrix} \alpha I & -I & 0 \\ 0 & \alpha I - b\Delta_N & -I \\ 0 & 0 & \alpha I - a\Delta_N \end{pmatrix} \quad \text{and} \quad A_2^N = \begin{pmatrix} -\alpha I & 0 & 0 \\ -c^2\Delta_N & -\alpha I & 0 \\ 0 & 0 & -\alpha I \end{pmatrix},$$

The operator $A_1^N : \mathcal{D}(A_1^N) \to X^N$ has maximal L_p-regularity on $\mathbb{R}_+$. This is seen as in the proof of Proposition 3.1 by considering $v_t + A_1^N v = F$, $v_0 = 0$ for $v = (v_1, v_2, v_3)^\top$ and $F = (f_1, f_2, f_3)^\top$ which explicitly reads

$$\begin{aligned} \partial_t v_1 + \alpha v_1 - v_2 &= f_1, & v_1(0) &= 0, \\ \partial_t v_2 + \alpha v_2 - b\Delta_N v_2 - v_3 &= f_2, & v_2(0) &= 0, \\ \partial_t v_3 + \alpha v_3 - a\Delta_N v_3 &= f_3, & v_3(0) &= 0. \end{aligned}$$

Again, for $F \in L_p(\mathbb{R}_+; X^N)$ one solves stepwise the equations above, starting with the last one. For the second equation, we need higher regularity for the heat equation provided by Lemma B.5. Since $A_2^N : X^N \to X^N$ is bounded, there exists some $\nu > 0$

such that $\nu + A_1^N + A_2^N = \nu + A^N$ has maximal L_p-regularity on $\mathbb{R}_+$. Maximal L_p-regularity of $\nu + A^{N,0}$ follows by considering the operators $A_1^{N,0}$ and $A_2^{N,0}$ which equal A_1^N and A_2^N upon replacement of Δ_N by $\Delta_{N,0}$ and proceeding as above. □

THEOREM 4.2. *Let $p \in (1, \infty)$ and define $\omega_0^N = \min\{a\lambda_1^N, b\lambda_1^N/2, c^2/b\}$.*

(i) The operator $A^N : \mathcal{D}(A^N) \to X^N$ has maximal L_p-regularity on finite time intervals $J = (0, T)$ and therefore

$$(\partial_t + A^N, \gamma_t) : W_p^1(J; X^N) \cap L_p(J; \mathcal{D}(A^N)) \to L_p(J; X^N) \times (X^N, \mathcal{D}(A^N))_{1-1/p,p}$$

is an isomorphism.

(ii) The operator $A^{N,0} : \mathcal{D}(A^{N,0}) \to X^{N,0}$ has maximal L_p-regularity on $\mathbb{R}_+$ and therefore

$$(\partial_t + A^{N,0}, \gamma_t) : \mathrm{e}^{-\omega}(W_p^1(J; X^{N,0}) \cap L_p(J; \mathcal{D}(A^{N,0}))) \to \mathrm{e}^{-\omega} L_p(J; X^{N,0}) \times (X^{N,0}, \mathcal{D}(A^{N,0}))_{1-1/p,p}$$

is an isomorphism whenever $\omega \in [0, \omega_0^N)$.

Proof. Since $\nu + A^N$ has maximal L_p-regularity on $\mathbb{R}_+$, multiplication of (4.2) with $\mathrm{e}^{-\nu t}$ shows that A^N has maximal L_p-regularity on bounded intervals $J = (0, T)$. Moreover, likewise to Lemma 3.2 we show $s(-A^{N,0}) = -\omega_0^N$ and follow the proof of Theorem 3.3 to obtain maximal L_p-regularity of $A^{N,0}$ on R_+. □

Theorem 4.2 immediately yields optimal regularity for (4.1) with homogeneous boundary conditions, i. e. $g = h = 0$.

COROLLARY 4.3. *Let $p \in (1, \infty)\backslash\{3\}$ and define $\omega_0^N = \min\{a\lambda_1^N, b\lambda_1^N/2, c^2/b\}$.*

(i) If $J = (0, T)$ is finite, then (4.1) with $g = h = 0$ admits a unique solution $u \in W_p^3(J; L_p(\Omega)) \cap W_p^1(J; W_p^4(\Omega))$ if and only if $f \in L_p(J \times \Omega)$, $u_0 \in W_p^4(\Omega)$, $u_1 \in W_p^{4-2/p}(\Omega)$, $u_2 \in W_p^{2-2/p}(\Omega)$ and the initial and boundary data are compatible.

(ii) If $J = \mathbb{R}_+$, then for every $\omega \in (0, \omega_0^N)$ we have that (4.1) with $g = h = 0$ admits a unique solution $u \in \mathrm{e}^{-\omega}\mathbb{E}_{u,0}$, where $\mathbb{E}_{u,0} = W_p^3(\mathbb{R}_+; L_{p,0}(\Omega)) \cap W_p^1(\mathbb{R}_+; W_p^4(\Omega) \cap L_{p,0}(\Omega))$ if and only if $f \in \mathrm{e}^{-\omega} L_p(\mathbb{R}_+; L_{p,0}(\Omega))$, $u_0 \in W_p^4(\Omega)$, $u_1 \in W_p^{4-2/p}(\Omega)$, $u_2 \in W_p^{2-2/p}(\Omega)$, where $u_0, u_1, u_2, \Delta u_0, \Delta u_1$ have zero mean, and the initial and boundary data are compatible.

Proof. The verification of the claim is done analogously to Corollary 3.4. □

Finally, we arrive at our optimal regularity result for (4.1).

PROPOSITION 4.4. *Let $p \in (1,\infty)\setminus\{3\}$ and define $\omega_0 = \min\{a\lambda_1^N, b\lambda_1^N/2, c^2/b\}$. Then for every $\omega \in (0,\omega_0)$ the linear initial boundary value problem (4.1) on $J = \mathbb{R}_+$ has a unique solution of the form $u(t,x) = v(t,x) + w(t)$, where*

$$v \in \mathrm{e}^{-\omega}\mathbb{E}_{u,0}, \quad \mathbb{E}_{u,0} = W_p^3(\mathbb{R}_+; L_{p,0}(\Omega)) \cap W_p^1(\mathbb{R}_+; W_p^4(\Omega) \cap L_{p,0}(\Omega)),$$
$$\partial_t^3 w \in \mathrm{e}^{-\omega} L_p(\mathbb{R}_+)$$

if and only if the data satisfy the conditions

(i) $f \in \mathrm{e}^{-\omega} L_p(\mathbb{R}_+ \times \Omega)$,
(ii) $u_0 \in W_p^4(\Omega)$, $u_1 \in W_p^{4-2/p}(\Omega)$, $u_2 \in W_p^{2-2/p}(\Omega)$,
(iii) $g \in \mathrm{e}^{-\omega}\mathbb{F}_{g,\nu}$, $\mathbb{F}_{g,\nu} = W_p^{5/2-1/2p}(\mathbb{R}_+, L_p(\Gamma)) \cap W_p^1(\mathbb{R}_+, W_p^{3-1/p}(\Gamma))$,
$h \in \mathrm{e}^{-\omega}\mathbb{F}_{h,\nu}$, $\mathbb{F}_{h,\nu} = W_p^{3/2-1/2p}(\mathbb{R}_+; L_p(\Gamma)) \cap W_p^1(\mathbb{R}_+; W_p^{1-1/p}(\Gamma))$,
(iv) *the data* (g, h, u_0, u_1, u_2) *are compatible.*

Moreover, the solution fulfills the estimate

$$\|u\|_{\mathrm{e}^{-\omega}\mathbb{E}_u} \lesssim \|f\|_{\mathrm{e}^{-\omega}L_p} + \|g\|_{\mathrm{e}^{-\omega}\mathbb{F}_{g,\nu}} + \|h\|_{\mathrm{e}^{-\omega}\mathbb{F}_{h,\nu}} + \|u_0\|_{W_p^4} + \|u_1\|_{W_p^{4-2/p}} + \|u_2\|_{W_p^{2-2/p}}.$$

Proof. Necessity and uniqueness can be proved similarly to the Dirichlet case, since apart from having zero mean v has the same regularity as u in Proposition 3.5. We point out that

$$\partial_\nu u_0|_\Gamma = g|_{t=0} \text{ in } W_p^{3-1/p}(\Gamma), \qquad \partial_\nu \Delta u_0|_\Gamma = h|_{t=0} \text{ in } W_p^{1-1/p}(\Gamma),$$
$$\partial_\nu u_1|_\Gamma = g_t|_{t=0} \text{ in } B_{p,p}^{3-3/p}(\Gamma), \qquad \partial_\nu \Delta u_1|_\Gamma = h_t|_{t=0} \text{ in } W_p^{1-3/p}(\Gamma) \text{ if } p > 3,$$
$$\partial_\nu u_2|_\Gamma = g_{tt}|_{t=0} \text{ in } W_p^{1-3/p}(\Gamma) \text{ if } p > 3.$$

It remains to show that conditions (i)–(iv) are sufficient for the existence of a solution $u(t,x) = v(t,x) + w(t)$ such that $v \in \mathrm{e}^{-\omega}\mathbb{E}_u$ and $w_{ttt} \in \mathrm{e}^{-\omega}L_p(\mathbb{R}_+)$. As in the Dirichlet case, we interchange the order of differentiation on the left-hand side and consider the subproblems

$$\begin{cases} \varphi_{tt} - b\Delta\varphi_t - c^2\Delta\varphi = f & \text{in } \mathbb{R}_+ \times \Omega, \\ \partial_\nu \varphi = ah - g_t & \text{on } \mathbb{R}_+ \times \Gamma, \\ (\varphi, \varphi_t) = (a\Delta u_0 - u_1, a\Delta u_1 - u_2) & \text{on } \{t=0\} \times \Omega, \end{cases} \tag{4.4}$$

and

$$\begin{cases} a\Delta u - u_t = \varphi & \text{in } \mathbb{R}_+ \times \Omega, \\ \partial_\nu u = g & \text{on } \mathbb{R}_+ \times \Gamma, \\ u = u_0 & \text{on } \{t=0\} \times \Omega, \end{cases} \tag{4.5}$$

From condition, (ii) we obtain $a\Delta u_0 - u_1 \in W_p^2(\Omega)$ and $a\Delta u_1 - u_2 \in W_p^{2-2/p}(\Omega)$. Furthermore, (iii) implies $ah - g_t \in \mathrm{e}^{-\omega}\mathbb{W}_\nu$. On the strength of Lemma 2.7, we obtain that (4.4) admits a unique solution of the form $\varphi(t,x) = \varphi_1(t,x) + \varphi_2(t)$ with $\varphi_1 \in \mathrm{e}^{-\omega}\mathbb{W}_{u,0}$ and $\partial_t^2\varphi_2 \in \mathrm{e}^{-\omega}L_p(\mathbb{R}_+)$. We now make the ansatz $u(x,t) = v(x,t) + w(t)$ such that $\bar{v}(\cdot,t) = 0$. Applying $|\Omega|^{-1}\int_\Omega$ to $a\Delta u - u_t = \varphi$ we deduce that w solves the ordinary differential equation $w_t = -\varphi_2 + a|\Gamma||\Omega|^{-1}\bar{g}$ with $w(0) = \bar{u}_0$. Hence, $\partial_t^3 w \in \mathrm{e}^{-\omega}L_p(\mathbb{R}_+)$. Moreover, v is a solution of

$$a\Delta v - v_t = \varphi_1 + a|\Gamma||\Omega|^{-1}\bar{g} \text{ in } \Omega, \quad \partial_\nu v = g \text{ on } \Gamma, \quad v(0) = u_0 - \bar{u}_0. \tag{4.6}$$

In order to apply Corollary B.4, we first note that the right-hand side $\varphi_1 + a|\Gamma||\Omega|^{-1}\bar{g}$ belongs to $\mathrm{e}^{-\omega}\mathbb{W}_u$ since $\bar{g}$ only depends on time and belongs to $e^{-\omega}W_p^2(\mathbb{R}_+)$. The rescaled function $v_a(t,x) = av(t/a,x)$ should solve the system

$$\Delta v_a - \partial_t v_a = \varphi_1 + a|\Gamma||\Omega|^{-1}\bar{g} \text{ in } \Omega, \quad \partial_\nu v_a = ag \text{ on } \Gamma, \quad v_a(0) = au_0 - a\bar{u}_0. \tag{4.7}$$

Hence, the compatibility condition (B.7) becomes $-\int_\Omega(\varphi_1 + a|\Gamma||\Omega|^{-1}\bar{g})\,dx + \int_\Gamma ag\,dS = 0$ and is clearly satisfied. Therefore, Corollary B.4 yields a unique solution $v_a \in \mathrm{e}^{-\omega}\mathbb{E}_{u,0}$ of problem (4.7), and thus, $u = v + w$ solves problem (4.1) on $J = \mathbb{R}_+$. □

REMARK 4.5. If we consider (4.1) on a finite time interval $J = (0,T)$ instead of $J = \mathbb{R}_+$, we may set $\omega = 0$ and obtain a unique solution $u \in W_p^3(J; L_p(\Omega)) \cap W_p^1(J; W_p^4(\Omega))$ if and only if conditions (i)–(iv) from Proposition 4.4 hold with $\omega = 0$ and $\mathbb{R}_+$ replaced by J.

4.2. Global well-posedness and exponential stability

We now at well-posedness for the Neumann problem (1.6). In case the (sufficiently small) data have nonzero mean, we only prove well-posedness of (1.6) on finite time intervals $J = (0,T)$. On the other hand, if the (sufficiently small) data have zero mean, then we obtain a globally well-posed solution which decays exponentially fast to zero.

THEOREM 4.6. (Well-posedness: the Neumann case) *Let $p > \max\{n/4 + 1/2, n/3\}$, $p \neq 3$, define $\omega_0 = \min\{a\lambda_1^N, b\lambda_1^N/2, c^2/b\}$ and suppose*

$$\begin{aligned} &u_0 \in W_p^4(\Omega), \qquad u_1 \in W_p^{4-2/p}(\Omega), \qquad u_2 \in W_p^{2-2/p}(\Omega),\\ &g \in \mathrm{e}^{-\omega}\mathbb{F}_{g,\nu}(J),\ \mathbb{F}_{g,\nu}(J) = W_p^{5/2-1/2p}(J, L_p(\Gamma)) \cap W_p^1(J, W_p^{3-1/p}(\Gamma)),\\ &h \in \mathrm{e}^{-\omega}\mathbb{F}_{h,\nu}(J),\ \mathbb{F}_{h,\nu}(J) = W_p^{3/2-1/2p}(J; L_p(\Gamma)) \cap W_p^1(J; W_p^{1-1/p}(\Gamma)), \end{aligned} \tag{4.8}$$

such that the data (g, h, u_0, u_1, u_2) are compatible.

(i) Let $J = (0,T)$ be finite and suppose (4.8) holds with $\omega = 0$. Then there exists some $\rho > 0$ such that if

$$\|g\|_{\mathbb{F}_{g,\nu}} + \|h\|_{\mathbb{F}_{h,\nu}} + \|u_0\|_{W_p^4} + \|u_1\|_{W_p^{4-2/p}} + \|u_2\|_{W_p^{2-2/p}} < \rho,$$

then problem (1.6) has a unique solution $u \in W_p^3(J; L_p(\Omega)) \cap W_p^1(J; W_p^4(\Omega))$.

(ii) *Let $J = \mathbb{R}_+$ and assume in addition that $u_0, u_1, u_2, \Delta u_0, \Delta u_1$ and g, h have zero mean. Then for every $\omega \in (0, \omega_0)$ there exists some $\rho > 0$ such that if*

$$\|g\|_{e^{-\omega}\mathbb{F}_{g,\nu}} + \|h\|_{e^{-\omega}\mathbb{F}_{h,\nu}} + \|u_0\|_{W_p^4} + \|u_1\|_{W_p^{4-2/p}} + \|u_2\|_{W_p^{2-2/p}} < \rho,$$

then problem (1.6) has a unique solution $u \in e^{-\omega}\mathbb{E}_{u,0}$, where $\mathbb{E}_{u,0} = W_p^3(\mathbb{R}_+; L_{p,0}(\Omega)) \cap W_p^1(\mathbb{R}_+; W_p^4(\Omega) \cap L_{p,0}(\Omega))$.

The solution depends analytically on the data (g, h, u_0, u_1, u_2) with respect to the corresponding topologies and the regularities of the data are necessary for the regularity of the solution.

Proof. In (i), we define $u_\star$ to be the solution according to Remark 4.5 which satisfies (4.1) for the data $(f = 0, g, h, u_0, u_1, u_2)$ and suppose $u_\bullet$ satisfies homogeneous boundary and initial conditions. The solution is then constructed as $u = u_\star + u_\bullet$ analogously to Theorem 3.6.

To obtain (ii), we define $u_\star$ to be the solution of (4.1) on $\mathbb{R}_+$ for $(f = 0, g, h, u_0, u_1, u_2)$ according to Proposition 4.4. This means that we actually have $u_\star(x,t) = v_\star(x,t) + w_\star(t)$, where $v_\star \in e^{-\omega}\mathbb{E}_{u,0}$ and $\partial_t^3 w_\star \in e^{-\omega}L_p(\mathbb{R}_+)$. However, an inspection the proof of Proposition 4.4, we see that $w_\star$ solves the ordinary differential equation $\partial_t w_\star(t) = -\varphi_2(t) + a|\Gamma||\Omega|^{-1}\bar{g}(t)$ with $w_\star(0) = \bar{u}_0$, where φ_2 comes from the solution $\varphi(x,t) = \varphi_1(x,t) + \varphi_2(t)$ of (4.4) and satisfies $\partial_t^2\varphi_2 \in e^{-\omega}L_p(\mathbb{R}_+)$. Our assumptions on u_0, u_1, u_2, g and h imply that $ah - g_t$, $a\Delta u_0 - u_1$ and $a\Delta u_1 - u_2$ have zero mean over Γ, respectively, Ω and since moreover here we have $f = 0$, the data of (4.4) have zero mean which, using Lemma 2.7, implies $\varphi_2(t) = 0$ for all $t \in \mathbb{R}_+$. We conclude $\partial_t w_\star(t) = 0$, $w_\star(0) = 0$; hence, $w_\star(t) = 0$ for all $t \in \mathbb{R}_+$. This means that $u_\star = v_\star$ under our assumptions and we may again follow the proof of Theorem 3.6. □

REMARK 4.7. In order to prove global well-posedness, we need to employ Proposition 4.4 for the linearized equation, where for given data $(f = 0, g, h, u_0, u_1, u_2)$ according to (ii)–(iv) the solution is of the form $u(t,x) = v(t,x) + w(t)$, where $v \in e^{-\omega}\mathbb{E}_{u,0}$ has zero mean and w is only time-dependent. If w does not vanish, we are not able to follow the proof of Theorem 3.6, since then the term $((u_t)^2)_{tt}$ in the nonlinear right-hand side of (1.6) causes problems. Recall (3.11) and note that due to Proposition 4.4 we in fact have $u_\star = v_\star + w_\star$ with $v_\star \in e^{-\omega}\mathbb{E}_{u,0}$ and $\partial_t^3 w_\star \in e^{-\omega}L_p(\mathbb{R}_+)$. Then $w_\star$, $\partial_t w_\star$ and $\partial_t^2 w_\star$ are in general not contained in $e^{-\omega}L_p(\mathbb{R}_+)$ and thus (3.12) fails. The assumptions in Theorem (4.6) ensure that w vanishes and well-posedness can be shown as in the Dirichlet case.

If the data have zero mean, we obtain exponential stability for the Neumann problem (1.6).

COROLLARY 4.8. (Exponential stability: the Neumann case) *Under the same assumptions as in Theorem 4.6*(ii)*, the solution u decays exponentially fast to zero as $t \to \infty$ in the sense that*

$$\|u(t)\|_{W_p^4} + \|u_t(t)\|_{W_p^{4-2/p}} + \|u_{tt}(t)\|_{W_p^{2-2/p}} \le C\mathrm{e}^{-\omega t}, \qquad t \ge 0,$$

for some $C \ge 0$ depending on the boundary and initial data g, h, u_0, u_1 and u_2.

Proof. Since $u \in \mathrm{e}^{-\omega}\mathbb{E}_{u,0} \hookrightarrow \mathrm{e}^{-\omega}\mathbb{E}_u$, the result follows likewise to Corollary 3.7. □

Appendix A. Temporal traces

In this section, we consider the temporal trace operator in some anisotropic fractional Sobolev spaces. We recall that a bounded linear operator $r\colon X \to Y$ between Banach spaces X and Y is called a *retraction*, if there is a bounded linear map $r^c\colon Y \to X$ such that $rr^c = I_Y$. Thus, r is surjective and r^c is a bounded right-inverse for r. The map r^c is called a co-retraction for r. The following trace theorem can be derived from [13, Lemma 11], [25, Section 2.2.1], [2, Proposition III.4.10.3].

THEOREM A.1. *Let A be the generator of a bounded analytic semigroup $(\mathrm{e}^{-tA})_{t\ge 0}$ in a Banach space X such that $A\colon \mathcal{D}(A) \to X$ has a bounded inverse, let $p \in (1,\infty)$ and let $\mathcal{D}_A(\alpha, p) = (X, \mathcal{D}(A))_{\alpha,p}$ for $\alpha \in (0,1)$ and $\mathcal{D}_A(1,p) = \mathcal{D}(A)$.*

Then for every $\alpha \in (1/p, 1]$, the trace operator

$$\gamma_t = \cdot|_{t=0}\colon W_p^\alpha(\mathbb{R}_+; X) \cap L_p(\mathbb{R}_+; \mathcal{D}_A(\alpha, p)) \to \mathcal{D}_A(\alpha - 1/p, p)$$

is a retraction, the operator

$$R_A\colon u_0 \mapsto (t \mapsto \mathrm{e}^{-tA}u_0),\ \ \mathcal{D}_A(\alpha - 1/p, p) \to W_p^\alpha(\mathbb{R}_+; X) \cap L_p(\mathbb{R}_+; \mathcal{D}_A(\alpha, p))$$

is a co-retraction for γ_t and the following embedding is continuous.

$$W_p^\alpha(\mathbb{R}_+; X) \cap L_p(\mathbb{R}_+; \mathcal{D}_A(\alpha, p)) \hookrightarrow BUC(\mathbb{R}_+; \mathcal{D}_A(\alpha - 1/p, p)).$$

This theorem can be applied to the spaces $W_p^\alpha(\mathbb{R}_+; W_p^s(\Omega; E)) \cap L_p(\mathbb{R}_+; W_p^{s+2\alpha}(\Omega; E))$ for $\alpha \in (1/p, 1]$ and $s \in [0,\infty)$, provided that $W_p^s(\Omega; E)$ and $W_p^{s+2\alpha}(\Omega; E)$ are both Bessel potential spaces or both Sobolev–Slobodeckij spaces. Here and in the following, we assume that E is a Banach space of class $\mathcal{HT}$ and has property (α), where we refer to [23] and [18] for more information. Examples of such spaces are the Hilbert spaces, the space $L_q(\Omega, \mathcal{A}, \mu; E)$ on a σ-finite measure space $(\Omega, \mathcal{A}, \mu)$ with $q \in (1,\infty)$, and the Sobolev–Slobodeckij spaces $W_p^s(\mathbb{R}^n; E)$, the Bessel potential spaces $H_p^s(\mathbb{R}^n; E)$, and the Besov spaces $B_{p,q}^s(\mathbb{R}^n; E)$ for $s \in (0,\infty)$ and $p, q \in (1,\infty)$. The properties $\mathcal{HT}$ and (α) are inherited to closed subspaces and isomorphic spaces.

Next, we indicate how Theorem A.1 can be applied. Let $\mathcal{E}_\Omega$ be an extension operator from Ω to $\mathbb{R}^n$, which acts as a bounded linear operator $W_p^t(\Omega; E) \to W_p^t(\mathbb{R}^n; E)$ for all $t \in [0, 2k]$. Such extension operators are defined in [1] for $t \in \mathbb{N}_0$ and their boundedness for $t \notin \mathbb{N}_0$ follows from real interpolation. Then it remains to study $\mathcal{E}_\Omega u$ in $\mathbb{R}_+ \times \mathbb{R}^n$, where the operator $A = 1 - \Delta$ in $X = W_p^s(\mathbb{R}^n; E)$ with domain

$W_p^{s+2}(\mathbb{R}^n; E)$ has the required properties. Indeed, [10, Section 5] covers the case $s = 0$ and an abstract result of Dore [15] covers the case $s \in (0, 2)\backslash\{1\}$. The remaining cases follow by means of isomorphic mappings, interpolation and by taking fractional powers.

Therefore, the temporal trace operator

$$\begin{aligned}&\gamma_t : u \mapsto ((\mathcal{E}_\Omega u)|_{t=0})|_\Omega,\\ &\qquad W_p^\alpha(\mathbb{R}_+; W_p^s(\Omega; E)) \cap L_p(\mathbb{R}_+; W_p^{s+2\alpha}(\Omega; E)) \to B_{p,p}^{s+2\alpha-2/p}(\Omega; E)\end{aligned}$$

is bounded. For the boundary spaces $W_p^\alpha(\mathbb{R}_+; W_p^s(\Gamma; E)) \cap L_p(\mathbb{R}_+; W_p^{s+2\alpha}(\Gamma; E))$, we use a common retraction $r\colon W_p^t(\mathbb{R}^{n-1}; E)^N \to W_p^t(\Gamma; E)$ for all $t \in [0, 2k]$ with some $N \in \mathbb{N}$. A co-retraction for r is constructed with a partition of unity for Γ and local parametrizations of Γ in the proof of Lemma B.6. Then the temporal trace operator can be written as $\gamma_t : u \mapsto r((r^c u)|_{t=0})$ and maps

$$W_p^\alpha(\mathbb{R}_+; W_p^s(\Gamma; E)) \cap L_p(\mathbb{R}_+; W_p^{s+2\alpha}(\Gamma; E)) \to B_{p,p}^{s+2\alpha-2/p}(\Gamma; E).$$

In order to construct functions with prescribed initial values, we consider an operator $A\colon \mathcal{D}(A) \subset X \to X$ as in Theorem A.1 and define the spaces

$$\mathcal{D}_A(k+\alpha, p) = A^{-k}\mathcal{D}_A(\alpha, p) = (\mathcal{D}(A^k), \mathcal{D}(A^{k+1}))_{\alpha,p}$$

for $k \in \mathbb{N}_0$, $\alpha \in [0, 1]$, $p \in (1, \infty)$. Then Theorem A.1 and the identity $\partial_y \mathrm{e}^{-yA} = -A\mathrm{e}^{-yA} = \mathrm{e}^{-yA}A$ yield the following result.

COROLLARY A.2. *Let $k \in \mathbb{N}_0$, $\alpha \in (1/p, 1]$ and $p \in (1, \infty)$. Then the operator*

$$\begin{aligned}&R_A\colon\ u \mapsto (t \mapsto \mathrm{e}^{-tA}u),\\ &\qquad \mathcal{D}_A(k+\alpha-1/p, p) \to W_p^{k+\alpha}(\mathbb{R}_+; X) \cap L_p(\mathbb{R}_+; \mathcal{D}_A(k+\alpha, p))\end{aligned}$$

is a bounded right-inverse for γ_t.

We next deal with higher-order initial conditions.

LEMMA A.3. *Let $\gamma_t^j = (\partial_t^j \cdot)|_{t=0}$ and let $l \in \mathbb{N}_0$, $m \in \mathbb{N}$ with $m \geq l+1$. Then the operator*

$$(\gamma_t^0, \gamma_t^1, \ldots, \gamma_t^l)\colon W_p^m(\mathbb{R}_+; X) \cap L_p(\mathbb{R}_+; \mathcal{D}(A^m)) \to \prod\nolimits_{j=0}^{l} \mathcal{D}_A(m-j-1/p, p)$$

is a retraction.

Proof. For $j \in \{0, 1, \ldots, l\}$ and $x \in X$, we define

$$(S_j^A x)(t, \cdot) = \sum\nolimits_{i=0}^{l} c_{ij}\mathrm{e}^{-t(1+i)A}A^{-j}x \quad \text{for } t \geq 0,$$

where, for each j, the $l+1$ numbers c_{ij} ($i \in \{0, 1, \ldots, l\}$) solve the linear system

$$\sum\nolimits_{i=0}^{l} c_{ij}(-(1+i))^m = \delta_{mj} \quad \text{for } m \in \{0, 1, \ldots, l\}.$$

It is straightforward to check that $(\partial_t^m S_j^A x)(0) = \delta_{mj} x$ for $m \in \{0, 1, \dots, l\}$. From Corollary A.2, we infer that the desired co-retraction is given by $S^A(x_0, x_1, \dots, x_l) = \sum_{j=0}^{l} S_j^A x_j$. □

Appendix B. Higher regularity for the heat equation

We study the regularity of solutions of the heat problem

$$\begin{cases} (\partial_t + \mu_B - \Delta)u = f & \text{in } J \times \Omega, \\ \gamma_B u = g & \text{on } J \times \Gamma, \\ u|_{t=0} = u_0 & \text{in } \Omega. \end{cases} \tag{B.1}$$

Here J is a bounded interval $(0, T)$ or the half line $(0, \infty)$ and Ω is a bounded domain in $\mathbb{R}^n$, $n \in \mathbb{N}$, with smooth boundary Γ. For $B \in \{D, N\}$, let μ_B be a real number and let

$$\gamma_D = \cdot|_\Gamma, \quad \gamma_N = (\partial_\nu \cdot)|_\Gamma = \nu \cdot (\nabla \cdot)|_\Gamma, \quad \gamma_t^j = (\partial_t^j \cdot)|_{t=0}, \; \gamma_t = \gamma_t^0$$

denote the Dirichlet, the Neumann and the temporal trace operators, respectively. Again we let $\lambda_0^D > 0$ denote the smallest eigenvalue of $-\Delta_D$ and $\lambda_1^N > 0$ denote the smallest nonzero eigenvalue of $-\Delta_N$. We will prove the following regularity result.

THEOREM B.1. *Let $B \in \{D, N\}$, $j_D = 0$, $j_N = 1$, $\mu_D \in (-\lambda_0^D, \infty)$, $\mu_N \in (0, \infty)$, $l \in \mathbb{N}_0$, $k \in \mathbb{N}$ and $p \in (1, \infty)$ such that $j_B/2 + 3/2p \neq 1$. Then problem (B.1) has a unique solution*

$$u \in \mathbb{E}^{l,k} = W_p^{l+k}(J; L_p(\Omega)) \cap W_p^l(J; W_p^{2k}(\Omega)), \tag{B.2}$$

if and only if the data (f, g, u_0) satisfy the regularity conditions

$$f \in \mathbb{E}^{l,k-1} = W_p^{l+k-1}(J; L_p(\Omega)) \cap W_p^l(J; W_p^{2k-2}(\Omega)), \tag{B.3a}$$

$$g \in \gamma_B \mathbb{E}^{l,k} = W_p^{l+k-j_B/2-1/2p}(J; L_p(\Gamma)) \cap W_p^l(J; W_p^{2k-j_B-1/p}(\Gamma)), \tag{B.3b}$$

$$u_0 \in \gamma_t \mathbb{E}^{l,k} = \begin{cases} W_p^{2k}(\Omega) & \text{if } l \geq 1, \\ W_p^{2k-2/p}(\Omega) & \text{if } l = 0, \end{cases} \tag{B.3c}$$

and the compatibility conditions

$$u_j = \gamma_t^{j-1} f + (\Delta - \mu_B) u_{j-1} \in \begin{cases} W_p^{2k}(\Omega) & \text{for } j \in \mathbb{N} \cap [1, l-1], \\ W_p^{2(l+k-j)-2/p}(\Omega) & \text{for } j \in \mathbb{N} \cap [l, l+k-1], \end{cases} \tag{B.4a}$$

$$\gamma_t^j g = \gamma_B u_j \quad \text{for } j \in \mathbb{N}_0, \; j \leq l + k - j_B/2 - 3/2p. \tag{B.4b}$$

REMARK B.2. (i) The space $\mathbb{E}^{0,1}$ ($l = 0$, $k = 1$) is the standard parabolic solution space.

(ii) If $l = 0$ and $J \times \Omega$ is the half space $\mathbb{R}_+ \times \mathbb{R}^n$ or the wedge $\mathbb{R}_+ \times \mathbb{R}^n_+$, then $\mathbb{E}^{0,k}$ is the anisotropic space $H_p^{2k/\boldsymbol{\nu}}(J \times \Omega)$ with weight $\boldsymbol{\nu} = (2, 1, \ldots, 1)$ in the sense of [3]. This fact will be used in the construction of functions with prescribed boundary values.

(iii) We exclude the case $j_B/2 + 3/2p = 1$ in order to avoid the more complicated trace spaces $\gamma_D W^{2/3}_{3/2}(\Omega)$ and $\gamma_N W^{4/3}_3(\Omega)$.

(iv) The additional regularity conditions (B.4a) follow from the nontriangular structure of the space $\mathbb{E}^{l,k}$ in the case $l \geq 1$ and are derived in Sect. B.1. For $j \geq l+1$, formula (B.4a) does not contain additional regularity conditions and should be merely understood as the definition of the functions u_j, which appear in (B.4b).

(v) Every solution satisfies the higher-order boundary conditions

$$\begin{aligned} &\gamma_B \Delta^{j+1} u = (\partial_t + \mu_B) g_j - \gamma_B \Delta^j f = g_{j+1} \\ &\quad \text{for } j \in \mathbb{N}_0 \cap [0, k-2], \quad \text{with } g_0 = g. \end{aligned} \tag{B.5}$$

With the temporal trace theorem and $u_i = \gamma_t^i u$, we obtain

$$\begin{aligned} &\gamma_B \Delta^j u_i = \gamma_t^i g_j \quad \text{for } j \in \mathbb{N}_0 \cap [0, k-1], \\ &\quad i \in \mathbb{N}_0 \cap [0, l + k - j - j_B/2 - 3/2p]. \end{aligned} \tag{B.6}$$

These equations are no additional regularity or compatibility conditions but follow from (B.4), (B.5), by induction over $j \in \mathbb{N}_0$.

Aiming at stability for Neumann boundary conditions, we will also prove the following result, where we consider the subspace $L_{p,0}(\Omega) = \{f \in L_p(\Omega) : \int_\Omega f(x) \mathrm{d}x = 0\}$.

COROLLARY B.3. *Let $B = N$, $\mu_N \in (-\lambda_1^N, \infty)$, $l \in \mathbb{N}_0$, $k \in \mathbb{N}$, $p \in (1, \infty)$, $p \neq 3$. Then problem (B.1) has a unique solution $u \in \mathbb{E}_0^{l,k} = \mathbb{E}^{l,k} \cap L_p(J; L_{p,0}(\Omega))$ if and only if the data (f, g, u_0) satisfy the regularity conditions (B.3) and the compatibility conditions (B.4) and*

$$\int_\Omega u_0(x)\, dx = 0, \quad \int_\Omega f(t, x)\, dx + \int_\Gamma g(t, x)\, dS(x) = 0 \textit{ for } t \in J. \tag{B.7}$$

Next, we study the original heat problem

$$\begin{cases} (\partial_t - \Delta) u = f & \text{in } \mathbb{R}_+ \times \Omega, \\ \gamma_B u = g & \text{on } \mathbb{R}_+ \times \Gamma, \\ u|_{t=0} = u_0 & \text{in } \Omega. \end{cases} \tag{B.8}$$

COROLLARY B.4. (i) *Let $B \in \{D, N\}$, $\mu_D \in (-\lambda_0^D, \infty)$, $\mu_N \in (0, \infty)$, $l \in \mathbb{N}_0$, $k \in \mathbb{N}$, $p \in (1, \infty)$ such that $j_B/2 + 3/2p \neq 1$. Then problem (B.8) has a unique solution $u \in \mathrm{e}^{\mu_B} \mathbb{E}^{l,k}$ if and only if the data (f, g, u_0) satisfy the regularity conditions*

$$(f, g, u_0) \in \mathrm{e}^{\mu_B}\mathbb{E}^{l,k-1} \times \mathrm{e}^{\mu_B}\gamma_B\mathbb{E}^{l,k} \times \begin{cases} W_p^{2k}(\Omega) & \text{if } l \geq 1, \\ W_p^{2k-2/p}(\Omega) & \text{if } l = 0, \end{cases} \tag{B.9}$$

and the compatibility conditions

$$u_j = \gamma_t^{j-1} f + \Delta u_{j-1} \in \begin{cases} W_p^{2k}(\Omega) & \text{for } j \in \mathbb{N} \cap [1, l-1], \\ W_p^{2(l+k-j)-2/p}(\Omega) & \text{for } j \in \mathbb{N} \cap [l, l+k-1], \end{cases} \tag{B.10a}$$

$$\gamma_t^j g = \gamma_B u_j \quad \text{for } j \in \mathbb{N}_0 \cap [0, l+k-j_B/2-3/2p]. \tag{B.10b}$$

(ii) *Let $B = N$, $\mu_N \in (-\lambda_1^N, \infty)$. Then problem* (B.8) *has a unique solution $u \in \mathrm{e}^{\mu_N}\mathbb{E}_0^{l,k}$ if and only if the data (f, g, u_0) satisfy the regularity conditions* (B.9) *and the compatibility conditions* (B.10), (B.7).

Proof. In problem (B.1) we multiply f, g with $\mathrm{e}^{\mu_B t}$, so that

$$\mathrm{e}^{\mu_B t} f = \mathrm{e}^{\mu_B t}(\partial_t + \mu_B - \Delta)u = (\partial_t - \Delta)\mathrm{e}^{\mu_B t}u, \quad \mathrm{e}^{\mu_B t} g = \gamma_B \mathrm{e}^{\mu_B t} u.$$

This shows that $\mathrm{e}^{\mu_B t}u$ solves (B.8) for $(\mathrm{e}^{\mu_B t} f, \mathrm{e}^{\mu_B t} g, u_0)$ if and only if u solves the shifted problem (B.1) for (f, g, u_0). Hence, Theorem B.1 and Corollary B.3 yield the assertions. □

B.1. Compatibility conditions

By means of the results from Sect. A, it is not difficult to verify that the regularity conditions (B.3) are indeed necessary for $u \in \mathbb{E}^{l,k}$. Let us now derive the remaining compatibility conditions.

First, any function $u \in \mathbb{E}^{l,k}$ satisfies the initial regularity conditions

$$\gamma_t^j u \in \begin{cases} W_p^{2k}(\Omega) & \text{for } j \in \mathbb{N}_0 \cap [0, l-1], \\ W_p^{2l+2k-2j-2/p}(\Omega) & \text{for } j \in \mathbb{N}_0 \cap [l, l+k-1]. \end{cases} \tag{B.11}$$

If u solves (B.1) with data (f, g, u_0), then an application of ∂_t^{j-1} to the heat equation yields

$$\partial_t^j u = \partial_t^{j-1} f + (\Delta - \mu_B)\partial_t^{j-1} u.$$

In particular, the initial values $u_j = \partial_t^j u|_{t=0}$ are given in terms of f and u_0.

Hence, (B.11) implies that the data (f, u_0) and u_j must satisfy

$$u_j = \gamma_t^{j-1} f + (\Delta - \mu_B)u_{j-1} \in \begin{cases} W_p^{2k}(\Omega) & \text{for } j \in \mathbb{N} \cap [1, l-1], \\ W_p^{2(l+k-j)-2/p}(\Omega) & \text{for } j \in \mathbb{N} \cap [l, l+k-1]. \end{cases}$$

For $j \in [1, l]$, this is indeed an additional condition, since f merely satisfies

$$\gamma_t^{j-1} f \in \begin{cases} W_p^{2k-2}(\Omega) & \text{for } j \in \mathbb{N}_0 \cap [0, l], \\ W_p^{2l+2k-2j-2/p}(\Omega) & \text{for } j \in \mathbb{N} \cap [l+1, l+k-1]. \end{cases}$$

The conditions for $l+1 \leq j \leq l+k-1$ then follow from the regularity of f, u_0, …, u_l and could therefore be omitted in (B.4a), but we keep them there as a definition of u_{l+1}, …, u_{l+k-1}. Indeed, these functions still admit traces on Γ. By differentiating the boundary condition $\gamma_B u = g$ with respect to time, we obtain

$$\gamma_t^j g = \gamma_B u_j \in \begin{cases} W_p^{2k-j_B-1/p}(\Gamma) & \text{for } j \in \mathbb{N}_0 \cap [0, l-1], \\ W_p^{2(l+k-j)-j_B-3/p}(\Gamma) & \text{for } j \in \mathbb{N}_0 \cap [l, l+k-j_B/2-3/2p]. \end{cases}$$

This shows that (B.4b) is a necessary condition. We conclude that the necessity part of Theorem B.1 is true; that is, if problem (B.1) has a solution $u \in \mathbb{E}^{l,k}$ with data (f, g, u_0), then (B.3) and (B.4) are satisfied. We next prepare the proof of the existence part.

B.2. Interior regularity and initial conditions

From [10, Theorem 8.2], we deduce that for $B \in \{D, N\}$ and $E \in \mathcal{HT}$ there exists $\mu_B \geq 0$ such that the realization $\mu_B - \Delta_B$ with domain $\mathcal{D}(\Delta_B) = \{u \in W_p^2(\Omega; E) : \gamma_B u = 0\}$ in $L_p(\Omega; E)$ has maximal regularity of type $L_p(\mathbb{R}_+; L_p(\Omega; E))$. Thus, the operator

$$\mu_B + \partial_t - \Delta_B : {}_0W_p^1(\mathbb{R}_+; L_p(\Omega; E)) \cap L_p(\mathbb{R}_+; \mathcal{D}(\Delta_B)) \to L_p(\mathbb{R}_+; L_p(\Omega; E))$$

is invertible for $B \in \{D, N\}$. By using [14, Theorem 2.4] and a spectral theoretic argument as in [26], we may even allow for $\mu_D \in (-\lambda_0^D, \infty)$, $\mu_N \in (0, \infty)$.

In order to obtain higher regularity results, we consider the spaces

$$X_B^k = (\mu_B - \Delta_B)^{-k} L_p(\Omega; E), \quad \|u\|_{X_B^k} = \|(\mu_B - \Delta_B)^k u\|_{L_p(\Omega; E)} \quad \text{for } k \in \mathbb{N}_0.$$

These spaces can be easily characterized by

$$X_B^k = \{u \in W_p^{2k}(\Omega; E) : \gamma_B \Delta^j u = 0 \text{ for } 0 \leq j \leq k-1\}.$$

By commuting the operator $\mu_B + \partial_t - \Delta_B$ with $(\mu_B - \Delta_B)^k$, it follows that $\mu_B - \Delta_B$ has maximal regularity of type $L_p(\mathbb{R}_+; X_B^k)$ for every $B \in \{D, N\}$, $k \in \mathbb{N}_0$, that is,

$$\mu_B + \partial_t - \Delta : {}_0W_p^1(\mathbb{R}_+; X_B^k) \cap L_p(\mathbb{R}_+; X_B^{k+1}) \to L_p(\mathbb{R}_+; X_B^k)$$

is a topological linear isomorphism. Moreover, the map $\epsilon + \partial_t : {}_0W_p^{l+1}(\mathbb{R}_+; E) \to {}_0W_p^l(\mathbb{R}_+; E)$ is a topological linear isomorphism for every $\epsilon > 0$ and every $l \in \mathbb{N}_0$, see e. g. [28]. Hence, by commuting $\mu_B + \partial_t - \Delta_B$ with $\epsilon + \partial_t$, we obtain the following result.

LEMMA B.5. *Let* $\mu_D \in (-\lambda_0^D, \infty)$, $\mu_N \in (0, \infty)$, $l \in \mathbb{N}_0$, $k \in \mathbb{N}_0$, $B \in \{D, N\}$. *Then the map*

$$\mu_B + \partial_t - \Delta_B \colon {}_0W_p^{l+1}(\mathbb{R}_+; X_B^k) \cap {}_0W_p^l(\mathbb{R}_+; X_B^{k+1}) \to {}_0W_p^l(\mathbb{R}_+; X_B^k)$$

is a topological linear isomorphism.

We next comment on function spaces for the initial data. From [3, Section 4.9], we obtain

$$\begin{aligned}(X_B^k, X_B^{k+1})_{1-1/p,p} = &\{u \in W_p^{2k+2-2/p}(\Omega; E) : \gamma_B \Delta^j u = 0 \\ &\text{for } 0 \le j \le k - j_B/2 - 3/2p\}.\end{aligned}$$

Then the temporal trace operator $\gamma_t \colon W_p^1(\mathbb{R}_+; X_B^k) \cap L_p(\mathbb{R}_+; X_B^{k+1}) \to (X_B^k, X_B^{k+1})_{1-1/p,p}$ is bounded and surjective, and therefore,

$$\begin{aligned}(\mu_B + \partial_t - \Delta_B, \gamma_t) \colon &W_p^1(\mathbb{R}_+; X_B^k) \cap L_p(\mathbb{R}_+; X_B^{k+1}) \\ &\to L_p(\mathbb{R}_+; X_B^k) \times (X_B^k, X_B^{k+1})_{1-1/p,p}\end{aligned}$$

is also a topological linear isomorphism for $B \in \{D, N\}$, $k \in \mathbb{N}_0$.

B.3. Boundary conditions

We will use the following result for constructing a function with prescribed boundary conditions (B.5).

LEMMA B.6. *Let* $l \in \mathbb{N}_0$, $m \in \mathbb{N}$, $k \in \mathbb{N}$, $p \in (1, \infty)$, *let* $\gamma_\nu^j = (\partial_\nu^j \cdot)|_\Gamma$ *in the sense of traces, and let*

$${}_0\mathbb{G}^{l,m/2}(\mathbb{R}_+ \times \Gamma) = {}_0W_p^{l+m/2-1/2p}(\mathbb{R}_+; L_p(\Gamma; E)) \cap {}_0W_p^l(\mathbb{R}_+; W_p^{m-1/p}(\Gamma; E)).$$

Then $\gamma_\nu^j : {}_0\mathbb{E}^{l,k}(\mathbb{R}_+ \times \Omega) \to {}_0\mathbb{G}^{l,k-j/2}(\mathbb{R}_+ \times \Gamma)$ *is a retraction and the operator*

$$\begin{aligned}\mathcal{B}_{l,k} = (\gamma_\nu^0 = \gamma_D, \gamma_\nu^1 = \gamma_N, \gamma_\nu^2, \dots, \gamma_\nu^{2k-1}) \colon &{}_0\mathbb{E}^{l,k}(\mathbb{R}_+ \times \Omega) \\ &\to \prod_{j=0}^{2k-1} {}_0\mathbb{G}^{l,k-j/2}(\mathbb{R}_+ \times \Gamma)\end{aligned}$$

is a retraction.

Proof. In the case $\Omega = \mathbb{R}_+^n$, $l = 0$, we infer from [3, Theorem 4.11.6] that

$$\overline{\mathcal{B}}_{0,k} = ((-1)^j \gamma_y^j)_{j=0}^{2k-1} : {}_0\mathbb{E}^{0,k}(\mathbb{R}_+ \times \mathbb{R}_+^n) \to \prod_{j=0}^{2k-1} {}_0\mathbb{G}^{0,k-j/2}(\mathbb{R}_+ \times \mathbb{R}^{n-1})$$

is a retraction. Let $\overline{\mathcal{B}}_{0,k}^c$ denote a co-retraction for $\overline{\mathcal{B}}_{0,k}$.

In the case $\Omega = \mathbb{R}_+^n$, $l \in \mathbb{N}_0$, we use the fact that $(\epsilon + \partial_t)^j : {}_0W_p^{s+j}(\mathbb{R}_+; F) \to {}_0W_p^s(\mathbb{R}_+; F)$ is invertible for every $\epsilon > 0$, $j \in \mathbb{N}$, $s \in [0, \infty)$ and every Banach space F of class $\mathcal{HT}$. Therefore, a co-retraction is given by $\mathcal{B}_{l,k}^c = (\epsilon + \partial_t)^{-l} \overline{\mathcal{B}}_{0,k}^c (\epsilon + \partial_t)^l$.

For bounded smooth domains, we define such operators by a localization technique. It is well known (see e. g. [16, Section 14.6], [29]) that the tubular neighborhood map

$$X\colon (x,t)\mapsto x+t\nu_\Gamma(x),\quad \Gamma\times(-R,R)\to B_R(\Gamma)=\{x\in\mathbb{R}^n : \operatorname{dist}(x,\Gamma)<R\}$$

is a homeomorphism for some $R>0$. Let $\{U_j : j\in I\}$ be a finite open covering of Γ in $\mathbb{R}^n$ and let $\{\varphi_j : j\in I\}\subset C_c^\infty(\Gamma)$ be a partition of unity subordinate to $\{U_j\cap\Gamma : j\in I\}$. Then there exists $r\in(0,R)$ such that $B_r(\Gamma)$ is covered by $\{U_j : j\in I\}$. For given $\chi\in C_c^\infty((-r,r))$ with $0\le\chi\le 1$ and $\chi(t)=1$ for $|t|\le r/2$, we extend φ_j to $\mathbb{R}^n$ by means of $\varphi_j(X(x,t))=\varphi_j(x)\chi(t)$ for $(x,t)\in\Gamma\times(-r,r)$ so that $\operatorname{supp}\varphi_j\subset U_j$ and $\partial_\nu^m\varphi_j=0$ near Γ for all $m\ge 1$.

In addition, let $U_j=B_r(x^{(j)})$ with $x^{(j)}\in\Gamma$ for some $r\in(0,R)$ and choose rigid transformations $\Xi_j\colon x\mapsto x^{(j)}+Q_jx$ with Q_j orthogonal such that $Q_j(-e_n)=\nu_\Gamma(x^{(j)})$. There exist $\omega_j\in C_c^\infty(\mathbb{R}^{n-1})$ with $\omega_j(0)=|\nabla\omega_j(0)|=0$ such that for $\theta_j(x',x_n)=(x',x_n+\omega_j(x'))$ we have $U_j\cap\Omega=U_j\cap\Xi_j(\theta_j(\mathbb{R}^n_+))$ and thus $U_j\cap\Gamma=U_j\cap\Xi_j(\theta_j(\Gamma_0))$ with $\Gamma_0=\mathbb{R}^{n-1}\times\{0\}$. Let us construct smooth diffeomorphisms Θ_j of $\mathbb{R}^n$ such that $U_j\cap\Omega=U_j\cap\Theta_j(\mathbb{R}^n_+)$ and $U_j\cap\Gamma=U_j\cap\Theta_j(\mathbb{R}^{n-1}\times\{0\})$. Given $r\in(0,R/2)$, $\psi\in C_c^\infty(B_{2r}(0))$ with $\psi=1$ on $B_r(0)$, let

$$\Theta_j(x)=\begin{cases}\psi(x)\left[\Xi_j(\theta_j(x',0))-x_n\nu_\Gamma(\Xi_j(\theta_j(x',0)))\right]+(1-\psi(x))\,\Xi_j(x) & \text{for } |x|\le 2r,\\ \Xi_j(x) & \text{for } |x|\ge 2r.\end{cases}$$

If $r\in(0,R/2)$ is sufficiently small, then Θ_j is a diffeomorphism since $\partial_x\Theta_j(x)\to Q_j$ as $r\to 0$, uniformly on $\mathbb{R}^n$. Moreover, Θ_j has the asserted properties and satisfies $-\partial_n\Theta_j(x',0)=\nu_\Gamma(\Theta_j(x',0))$ and $\partial_n^m\Theta_j(x',0)=0$ for all $m\ge 2$ and $x'\in B_r(0)$.

Choose smooth cutoff functions $\psi_j\in C_c^\infty(\Theta_j^{-1}(U_j))$ with $\psi_j=1$ on $\Theta_j^{-1}(\operatorname{supp}\varphi_j)$ and define the multiplication operator $M_j\colon u\mapsto\psi_j u$. With the pullback $\Theta_j^*\colon u\mapsto u\circ\Theta_j$ and the push forward $\Theta_{j*}\colon u\mapsto u\circ\Theta_j^{-1}$, we define a co-retraction for $\mathcal{B}_{l,k}$ by

$$\mathcal{B}_{l,k}^c g=\sum\nolimits_{j\in I}\Theta_{j*}M_j\overline{\mathcal{B}}_{l,k}^c\Theta_j^*(\varphi_j g)\quad\text{for } g\in\prod\nolimits_{j=0}^{2k-1}{}_0\mathbb{G}^{l,k-j/2}(\mathbb{R}_+\times\Gamma).$$

By means of the chain rule, Hölder's inequality and the mixed derivative embeddings, it can be shown that the linear operators $g\mapsto\varphi_j g$, Θ_j^*, M_j and Θ_{j*} act continuously in the relevant spaces and the properties of Θ_j and φ_j with respect to the normal direction imply that indeed $\mathcal{B}_{l,k}\mathcal{B}_{l,k}^c g=g$. This concludes the proof of Lemma B.6. □

B.4. Proof of Theorem B.1

It remains to prove the uniqueness and existence of a solution $u\in\mathbb{E}^{l,k}$ for given data (f,g,u_0). For proving uniqueness, it suffices to consider the most general case $l=0$, $k=1$, where $\mathbb{E}^{l,k}=W_p^1(J;L_p(\Omega))\cap L_p(J;W_p^2(\Omega))$ and $(f,g,u_0)=0$. If further μ_0 is sufficiently large, then the general result of [10] implies that $\mu_0-\Delta$ has maximal regularity of type $L_p(\mathbb{R}_+;L_p(\Omega))$ and this yields $u=0$ in case $\mu_B\ge\mu_0$.

Next, we employ spectral theory to cover the case $\mu_B \in (-\lambda_0^B, \infty)$, where $\lambda_0^B = \lambda_0(-\Delta_B) \geq 0$ denotes the smallest eigenvalue of $-\Delta_B$. It is well known that, since $\mathcal{D}(\Delta_B)$ is compactly embedded into $L_p(\Omega)$, the spectrum of Δ_B is discrete and consists only of eigenvalues with finite multiplicity. The eigenfunctions belong to

$$\mathcal{D}(\Delta_B^m) = \left\{u \in W_p^{2m}(\Omega) : \gamma_B(\Delta^j u)|_\Gamma = 0 \text{ on } \Gamma \text{ for } j \leq m-1\right\}$$

for every $m \in \mathbb{N}$ (see the proof of [5, Lemma 4.5]) and hence belong to $W_p^2(\Omega)$ and an integration by parts implies that the spectrum of Δ_B is contained in $(-\infty, -\lambda_0^B]$. A result of Dore [14, Theorem 2.4] implies that $\mu_B - \Delta_B$ has maximal regularity of type $L_p(\mathbb{R}_+; L_p(\Omega))$ for each $\mu_B \in (-\lambda_0^B, \infty)$ and this ensures uniqueness.

Existence. We construct a solution $u = u^1 + u^2 + u^3 \in \mathbb{E}^{l,k}$ such that

$$\begin{aligned}
&(\partial_t + \mu_B - \Delta)u^1 =: f^1, && && \gamma_t^i u^1 = u_i, \\
&(\partial_t + \mu_B - \Delta)u^2 =: f^2, && \gamma_B \Delta^j u^2 = g_j - \gamma_B \Delta^j u^1, && \gamma_t^i u^2 = 0, \\
&(\partial_t + \mu_B - \Delta)u^3 = f - f^2 - f^1, && \gamma_B \Delta^j u^3 = 0, && \gamma_t^i u^3 = 0,
\end{aligned}$$

for all i, j with $0 \leq i \leq l+k-1$ and $0 \leq j \leq k-1$.

Here the functions u_i and g_j are defined according to (B.4a) and (B.5) by

$$\begin{aligned}
u_i &= \gamma_t^{i-1} f + (\Delta - \mu_B)u_{i-1} && \text{for } i \in \mathbb{N} \cap [1, l+k-1], \\
g_j &= -\gamma_B \Delta^{j-1} f + (\partial_t + \mu_B) g_{j-1} && \text{for } j \in \mathbb{N} \cap [1, k-1],\ g_0 = g.
\end{aligned}$$

Construction of u^1. Let r_Ω^c be a common co-retraction for the restriction $r_\Omega \colon W_p^t(\mathbb{R}^n) \to W_p^t(\Omega)$ for all $t \in [0, 2k]$ (cf. [1, Theorem 5.22]). With the co-retraction S^A for the operator $(\gamma_t^0, \ldots, \gamma_t^{l+k-1})$ from Lemma A.3, we define

$$\begin{aligned}
u^1 = {}& r_\Omega S^I(r_\Omega^c u_0, r_\Omega^c u_1, \ldots, r_\Omega^c u_{l-1}, 0, \ldots, 0) \\
&+ r_\Omega S^{I-\Delta}(0, \ldots, 0, r_\Omega^c u_l, \ldots, r_\Omega^c u_{l+k-1}).
\end{aligned}$$

Here we consider the identity operator $I \colon \mathcal{D}(I) \to X$ with $X = \mathcal{D}(I) = W_p^{2k}(\mathbb{R}^n)$ so that the first summand of u^1 belongs to $(t \mapsto \mathrm{e}^{-t}) BUC^\infty(\mathbb{R}_+; W_p^{2k}(\Omega)) \hookrightarrow \mathbb{E}^{l,k}$. In the second summand, we consider the operator $I - \Delta \colon \mathcal{D}(\Delta) \to X$ in $X = L_p(\mathbb{R}^n)$ with domain $\mathcal{D}(\Delta) = W_p^2(\mathbb{R}^n)$ so that $r_\Omega^c u_l \in \mathcal{D}_\Delta(k - 1/p, p)$ and thus $S^{I-\Delta}(0, \ldots, 0, r_\Omega^c u_l, \ldots, r_\Omega^c u_{l+k-1}) \in W_p^{l+k}(\mathbb{R}_+; L_p(\mathbb{R}^n)) \cap L_p(\mathbb{R}_+; W_p^{2l+2k}(\mathbb{R}^n))$.

Construction of u^2. From (B.6) it follows that $\gamma_t^i g_j - \gamma_B \Delta^j u_i = 0$ for all $i, j \in \mathbb{N}_0$ with $j \leq k-1, i \leq l+k-j-1$. Thus, $g_j - \gamma_B \Delta^j u^1$ belongs to $\gamma_{B0}\mathbb{E}^{l,k-j}$. Near Γ we can split the Laplacian into $\Delta = \Delta_\Gamma + H_\Gamma \partial_\nu + \partial_\nu^2$ with the Laplace–Beltrami operator $\Delta_\Gamma = \operatorname{div}_\Gamma \nabla_\Gamma$ and some $H_\Gamma \in C^\infty(\Gamma)$. The operator Δ_Γ commutes with ∂_ν since it only depends on tangential derivatives. Therefore, the normal traces $h_j = (\partial_\nu^j u^2)|_\Gamma$ ($j \in \{0, \ldots, 2k-1\}$) of the desired solution $u^2 \in {}_0\mathbb{E}^{l,k}$ are uniquely determined by requiring that $h_{2j+j_B+1} = 0$ and

$$\gamma_B(\Delta_\Gamma + H_\Gamma \partial_\nu + \partial_\nu^2)^j u^2 = g_j - \gamma_B \Delta^j u^1 \quad \text{for } 0 \leq j \leq k-1.$$

With the co-retraction $\mathcal{B}^c_{l,k}$ from Lemma B.6, we define $u^2 = \mathcal{B}^c_{l,k}(h_0, \dots, h_{2k-1})$.

Construction of u^3. The compatibility conditions yield $\gamma_B \Delta^j f = 0$ for $0 \le j \le k-2$ and

$$f^3 = f - f^1 - f^2 \in {}_0W_p^{l+k-1}(J; L_p(\Omega)) \cap {}_0W_p^l(J; \mathcal{D}(\Delta_B^{k-1})).$$

Hence, $u^3 = (\partial_t + \mu_B - \Delta)^{-1} f^3$ is well defined by Lemma B.5.

Proof of Corollary B.3. With the same arguments as above and using the spectral properties of $-\Delta_{N,0}$, we see that $\mu_N - \Delta_N$ has maximal regularity of type $L_p(\mathbb{R}_+; L_{p,0}(\Omega))$. Hence, problem (B.1) has at most one solution within the space $\mathbb{E}_0^{l,k}$. Analogously as for Lemma B.5, we conclude that

$$\begin{aligned}\partial_t + \mu_N - \Delta_N \colon {}_0W_p^{l+k}(\mathbb{R}_+; L_{p,0}(\Omega)) \cap {}_0W_p^l(\mathbb{R}_+; \mathcal{D}(\Delta_N^k)) \\ \to {}_0W_p^{l+k-1}(\mathbb{R}_+; L_{p,0}(\Omega)) \cap {}_0W_p^l(\mathbb{R}_+; \mathcal{D}(\Delta_N^{k-1}))\end{aligned}$$

is an isomorphism. For proving existence, we modify the above representation $u = u^1 + u^2 + u^3$. For $i \in \{1, 2\}$, we replace u^i by $u^i - \bar{u}^i \in \mathbb{E}_0^{l,k}$, since $\partial_\nu \bar{u}^i = 0$ and $\bar{u}^i(0) = \bar{u}_0 = 0$. Then $\bar{f}^3(t) = \bar{f}(t) + |\Gamma||\Omega|^{-1}\bar{g}(t) = 0$ and thus $u^3 = (\partial_t + \mu_N - \Delta)^{-1} f^3$ belongs to $\mathbb{E}_0^{l,k}$. □

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